18.726: ALGEBRAIC GEOMETRY II, SPRING 2020

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1. 04/02/2020

1.1. **Cohomology.**

Proposition 1.1. *For any ringed space $(X, \mathcal{O}_X)$, the category of $\mathcal{O}_X$-modules has enough injectives.*

Proof. Find such an injective for each stalk $\mathcal{F}_x \hookrightarrow I_x$, and if $j_x\colon (x, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$, we consider $I = \prod (j_x)_* I_x$, and then $\mathrm{Hom}(\mathcal{G}, I) = \prod \mathrm{Hom}(\mathcal{G}, I_x) \simeq \mathrm{Hom}(\mathcal{G}_x, I_x)$, which is exact. □

Definition 1.2. Sheaf cohomology $H^i(X, -)$ is the right derived functor of $\Gamma(X, -)$ in the category of sheaf of abelian groups.

Remark 1.3. If $\mathcal{F}$ has extra structure, i.e. an $\mathcal{O}_X$-module, it is not a priori clear that one can take the derived functor inside the category. For this we need to prove injective objects inside the smaller category are acyclic.

Lemma 1.4. *If I is injective $\mathcal{O}_X$-module, then it is flasque.*

Proof. For $V \subseteq U$, we have $0 \to i_! \mathcal{O}_V \to \mathcal{O}_U$, and taking homs we get $I(U) \twoheadrightarrow I(V)$. □

Proposition 1.5. *Flasque sheafs are acyclic.*

Proof. If $\mathcal{F}$ is flasque, take $0 \to \mathcal{F} \to I \to \mathcal{G} \to 0$. Then by an exercise before, we have $\mathcal{G}$ is also flasque, and from the long exact sequence, we have $H^1(\mathcal{F}) = 0$ and $H^{i+1}(\mathcal{F}) \simeq H^i(\mathcal{G})$ for $i \geq 1$. Since $\mathcal{G}$ is also flasque, this inductively show $H^i(\mathcal{F}) = 0$ for $i > 0$. □

Corollary 1.6. *The sheaf cohomology can be taken as the derived functor inside the category of $\mathcal{O}_X$-modules.*

Remark 1.7. If $\mathcal{F}$ is a $\mathcal{O}_X$-module, then the cohomology $H^i(X, \mathcal{F})$ are $\Gamma(X, \mathcal{O}_X)$-modules.

Theorem 1.8 (Grothendieck)**.** *If X is a Noetherian topological space, then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.*

2. 06/02/2020

We first do some preparation for proving the theorem.

Lemma 2.1. *Direct limits of flasque sheafs in a Noetherian space is flasque.*

Proof. This is because for a Noetherian space, the presheaf $U \mapsto \varinjlim_{\alpha} \mathcal{F}_\alpha(U)$ is already a sheaf. □

Theorem 2.2. *For a Noetherian space cohomology commutes with direct limits.*

Proof. The map $\varinjlim_\alpha H^i(X, \mathcal{F}_\alpha) \to H^i(X, \varinjlim_\alpha \mathcal{F}_\alpha)$ is natural. We know it is an isomorphism for $i = 0$, and we think of both sides as a δ-functor from the category of directed systems of sheafs. Since the right side is a derived functor, we only need to show that they are effaceable. To do this, consider $\mathcal{G}_\alpha$ by $\mathcal{G}_\alpha(U) = \{s \to \bigcup_{t\in U} (\mathcal{F}_\alpha)_t,\ s_t \in \mathcal{F}_\alpha(t)\}$. This forms a directed system, and is it flasque by the above, and this proves that both sides are effaceable. □

Proof of Serre's Vanishing Theorem. For $i\colon Y \subseteq X$ closed and $j\colon U = X - Y \subseteq X$, we denote $\mathcal{F}_Y = i_*(\mathcal{F}|_Y)$ and $\mathcal{F}_U = i_!(\mathcal{F}|_U)$. Then we have the exact sequence $0 \to \mathcal{F}_u \to \mathcal{F} \to \mathcal{F}_Y \to 0$, and the game is to consider the analogous sequence for $\overline{U}$. We will use that $H^i(Y, \mathcal{F}|_Y) = H^i(X, \mathcal{F}_Y)$, which is easy since i_* preserves flasqueness.

If X is not irreducible, then let Y be a irreducible component of X, so that $\overline{U} \neq X$. So by Noetherian induction, it suffices to prove for the case that X is irreducible.

If $\dim X = 0$, then it has the indiscrete topology, so any sheaf is flasque.

For α a finite set of sections of $\mathcal{F}$ (not global sections), we consider $\mathcal{F}_\alpha$ the sheaf generated by such sections, namely $\mathcal{F}_\alpha = \sum_{(s\in\mathcal{F}(U))\in\alpha} i_*(\mathbb{Z}\cdot s)$. Let A be the collection of such α, which form a directed set. Then $\mathcal{F} = \varinjlim_\alpha \mathcal{F}_\alpha$. So it suffices to prove the theorem for such $\mathcal{F}_\alpha$, and if $\alpha' \leq \alpha$, the quotient $\mathcal{F}_\alpha/\mathcal{F}_{\alpha'}$ is generated by $\#\alpha - \#\alpha'$ sections. So by induction we only need to prove the theorem for sheafs generated by one section.

So $\mathcal{F}$ fits into a sequence $0 \to \mathcal{G} \to \mathbb{Z}_U \to \mathcal{F} \to 0$. Since X is irreducible, we can find $V \subseteq U$ open such that $\mathcal{G}|_V = \mathbb{Z}_V \xrightarrow{d} (\mathbb{Z}_U)|_V$ (choose d to be the smallest d on all stalks). So now $0 \to \mathbb{Z}_V \to \mathcal{G}$ and the cokernel is supported in a set of smaller dimension. Hence we reduced the problem to the sheafs $\mathbb{Z}_U$.

Going back to $0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0$, But know $\mathbb{Z}$ is flasque since X is irreducible, and so the result follows from induction and from the long exact sequence. □

2.1. **Cohomology of Noetherian affine scheme.** For A noetherian, and I a module. For an ideal $\mathfrak{a}$, we define $J = \Gamma_{\mathfrak{a}}(I) = I[\mathfrak{a}^\infty]$.

Proposition 2.3. *If I is injective, then J is injective.*

Proof. We need to check that if $\mathfrak{b}$ is an ideal and $\varphi\colon \mathfrak{b} \to J$, then we can lift to A. Since $\mathfrak{b}$ is finitely generated, there is n with $\mathfrak{a}^n\varphi(\mathfrak{b}) = 0$. By Krull's theorem, we can find n' with $\mathfrak{a}^{n'} \cap \mathfrak{b} \subseteq \mathfrak{a}^n\mathfrak{b}$. Then φ factors through $\mathfrak{b} \to \mathfrak{b}/(\mathfrak{a}^{n'} \cap \mathfrak{b})$. Since I is injective, we can extend the map to $A/\mathfrak{a}^{n'} \to I$, and then we know that this lands in J, so that J is injective. □

Lemma 2.4. *If A Noetherian and I injective, then $\theta\colon I \to I_f$ is surjective for any $f \in A$.*

Proof. Let $\mathfrak{b}_i = \operatorname{Ann}(f^i)$. Since A is Noetherian, this stabilizes, say at r. For $x \in I_f$, write $x = \theta(y)/f^n$. We may define $\varphi\colon (f^{n+r}) \to I$ by $f^{n+r}a \mapsto f^r ay$. This is well defined since the annihilator of f^{n+r} is $\mathfrak{b}_r$, which kills $f^r y$. Since I is injective, this extends to $A \to I$, and so if 1 maps to z, we have $f^{n+r}z = f^r y$, and so $\theta(z) = x$. □

Proposition 2.5. *Let A Noetherian and I injective. then $\tilde{I}$ is flasque.*

Proof. Let $U \subseteq X$ open, and let $Y = \operatorname{supp}\tilde{I}$. If $Y \cap U = 0$, $\tilde{I}(U) = 0$. If not, then choose $X_f \subseteq U$ such that $X_f \cap Y \neq \emptyset$. From the above, $\Gamma(X, \tilde{I}) \to \Gamma(U, \tilde{I}) \to \Gamma(X_f, \tilde{I})$ is surjective. For $Z = X - X_f$, and $s \in \Gamma(U, \tilde{I})$, we may subtract from an element of $\Gamma(X, \tilde{I})$ to assume $s \in \Gamma_Z(U, \tilde{I})$. So it suffices to find a lift in $\Gamma_Z(X, \tilde{I})$. Letting $J = \Gamma_f(I)$, it is injective, and by Noetherian induction we may assume $\tilde{J}$ is flasque. And now the claim follows from noting that $\Gamma_Z(U, \tilde{I}) = \Gamma(U, \tilde{J})$. □

3. 11/02/2020

Corollary 3.1. *If X is Noetherian and $\mathcal{F}$ is quasi-coherent, then $\mathcal{F}$ can be embedded into a quasi-coherent flasque sheaf.*

Proof. Write $X = \bigcup_{i \in I} U_i$ affine $U_i = \operatorname{Spec} A_i$ for I finite. Then $\mathcal{F}|_{U_i} = \tilde{M}_i$, and we then consider $0 \to \mathcal{F} \to \bigoplus j_*(\tilde{I}_i)$ for $M_i \hookrightarrow I_i$ injective modules. □

Theorem 3.2 (Serre)**.** *Let X be Noetherian. Then the following are equivalent: (i) X is affine, (ii) $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and quasi-coherent $\mathcal{F}$, (iii) $H^1(X, \mathcal{I}) = 0$ for all ideal sheafs $\mathcal{I}$.*

Proof. That (i) $\implies$ (ii) is trivial from the previous corollary. So it suffices to prove (iii) $\implies$ (i).

For a closed point $p \in X$, take an affine $p \in U$. Then we have $0 \to I_{p \cup Z} \to I_Z \to k(p) \to 0$. Then by (iii) we have $\Gamma(I_Z) \twoheadrightarrow \Gamma(k(p))$. Take $f \mapsto 1$. Then $X_f \subseteq U$, and contains p, and so X_f is affine. Then there is finitely many f_i such that X_{f_i} cover X. By a previous exercise, it suffices to check that $f_i \in \Gamma(X, \mathcal{O}_X)$ generate the unit ideal. Now consider

$$0 \to \mathcal{F} \to \mathcal{O}_X^n \to \mathcal{O}_X \to 0$$

and take global sections. We need to prove $H^1(X, \mathcal{F}) = 0$. We do this by considering the filtration $F \cap \mathcal{O}^i$, and then each subquotient is an ideal sheaf. □

3.1. **Cech Cohomology.** For a covering U_i of X, consider $U_{i_0,\dots,i_p} = \bigcap U_{i_j}$, with the convention that it is 0 is two are the same, and is alternating with respect to the order. We denote $C^p(U,\mathcal{F}) = \bigoplus \mathcal{F}(U_{i_0,\dots,i_p})$ the chains, with differentials given by

$$(d\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,\hat{i_k},\dots,i_{p+1}}$$

Definition 3.3. The Cech cohomology with respect to the covering U is $H^*(U,\mathcal{F}) = H^*(C^*(U,\mathcal{F}))$.

Remark 3.4. $H^*(U,\cdot)$ is not in general a δ-functor, but for any cover, $\Gamma(X,\mathcal{F}) = H^0(U,\mathcal{F})$.

Now we sheafify the construction of Cech cohomology: Let $\mathcal{C}^p = \prod f_*(\mathcal{F}|_{U_{i_0,\dots,i_p}})$, and consider the same differentials.

Lemma 3.5. $0 \to \mathcal{F} \to \mathcal{C}^\bullet$ *is exact.*

Proof. This is easy to check on the level of stalks: we define $k\colon \mathcal{C}^p_x \to \mathcal{C}^{p-1}_x$ by chosing $x \in V \subseteq U_{j_0}$ and defining $(k\alpha)_{i_0,\dots,i_{p-1}} = \alpha_{j_0,i_0,\dots,i_{p-1}}$. Then $dk + kd = 1$, and so it is exact. □

Corollary 3.6. *If $\mathcal{F}$ is flasque, then $H^p(U,\mathcal{F}) = 0$ for $p > 0$ and any cover.*

Corollary 3.7. *This is because $\mathcal{C}^i$ are also flasque.*

Lemma 3.8. *Let X be a topological space with a cover U. Then there is a natural functor $H^p(U,\mathcal{F}) = H^p(X,\mathcal{F})$.*

Proof. Just take a lift from $0 \to \mathcal{F} \to \mathcal{C}^\bullet$ to $0 \to \mathcal{F} \to \mathcal{I}^\bullet$. □

Theorem 3.9. *Let X be separated Noetherian (not really needed). Let U_i be an affine cover of X. Then $H^i(U,\mathcal{F}) = H^i(X,\mathcal{F})$ for $\mathcal{F}$ quasi-coherent.*

Proof. Let $\mathcal{G}$ be flasque with $0 \to \mathcal{F} \to \mathcal{G}$. Then we get $H^i(\mathcal{G}/\mathcal{F}) \simeq H^{i+1}(\mathcal{F})$ for $i \geq 1$, and since the covering is affine and X is separated, we have $0 \to C^\bullet(\mathcal{F}) \to C^\bullet(\mathcal{G}) \to C^\bullet(\mathcal{G}/\mathcal{F}) \to 0$, and so we also have a long exact sequence. Then comparing the two we get the isomorphism we wanted by induction. □

4. 25/02/2020

Last time we saw that for $\mathbb{P}^n$, $\omega_{\mathbb{P}^n}$ worked for duality.

Definition 4.1. We say ω_X^0 on a proper X over k is a dualizing sheaf if there is $H^n(X,\omega_X^0) \to k$ such that for all coherent sheafs $\mathcal{F}$ on X we have perfect pairings

$$\operatorname{Hom}(\mathcal{F},\omega_X^0) \times H^n(X,\mathcal{F}) \to H^n(X,\omega_X^0) \to k$$

Proposition 4.2. *If a dualizing sheaf exists, then it is unique.*

Proof. If we had two ω_X^0, $\omega_X^{0'}$, then

$$\operatorname{Hom}(\omega_X^0,\omega_X^{0'}) \times H^n(X,\omega_X^{0'}) \to H^n(X,\omega_X^0) \to k$$

Then the map $H^n(X,\omega^{0'}) \to k$ corresponds to an element $\Phi \in \operatorname{Hom}(\omega_X^0,\omega_X^{0'})$. In the same way we get Φ', and we can check their composite is the identity. □

We want to prove this exists. It exists in the proper case, but we will only prove in the projective case. We will work to prove the following.

Theorem 4.3. *If X is projective of dimension n over a field k, then $\omega_X^0 = \mathscr{E}xt^{N-n}(\mathcal{O}_X,\omega_{\mathbb{P}^N})$ is a dualizing sheaf where $X \hookrightarrow \mathbb{P}^N$.*

Lemma 4.4. *We have $\mathscr{E}xt^i(\mathcal{O}_X,\omega_{\mathbb{P}^N}) = 0$ for $i < N-n$.*

Proof. For a large enough, $\Gamma(\mathscr{E}^i(\mathcal{O}_X,\omega_{\mathbb{P}^N})(a)) = \operatorname{Ext}^i(\mathcal{O}_X,\omega_{\mathbb{P}^N}(a))$, and by duality on $\mathbb{P}^N$, this is $H^{N-i}(\mathbb{P}^N,\mathcal{O}_X(-a))^*$, which is 0 if $i < N - \dim X$. □

Lemma 4.5. *Let $\omega_X^0 = \mathscr{E}xt^{N-n}(\mathcal{O}_X,\omega_{\mathbb{P}^N})$. Then there is a functorial isomorphism $\operatorname{Hom}(\mathcal{F},\omega_X^0) \xrightarrow{\sim} \operatorname{Ext}^r(\mathcal{F},\omega_{\mathbb{P}^N})$ where $r = N-n$.*

Proof. Take an injective resolution $0 \to \omega_{\mathbb{P}^N} \to \mathcal{F}^\bullet$. Let $\mathscr{I}^\bullet = \mathscr{H}om(\mathcal{O}_X,\mathcal{F}^\bullet)$. Then $\operatorname{Hom}_{\mathbb{P}^N}(\mathcal{F},\mathcal{F}^i) = \operatorname{Hom}_X(\mathcal{F},\mathscr{I}^i)$. Then $\mathscr{E}xt^i(\mathcal{F},\omega_{\mathbb{P}^N}) = h^i(\operatorname{Hom}_{\mathbb{P}^N}(\mathcal{F},\mathcal{F}^\bullet)) = h^i(\operatorname{Hom}_X(\mathcal{F},\mathscr{I}^\bullet))$.

Taking $\mathcal{F} = \mathcal{O}_X$, from the previous lemma we have that $h^i(\mathscr{I}^\bullet)$ is 0 for $i < r$. So $I^\bullet = I_1^\bullet \oplus I_2^\bullet$ with $I_1^\bullet$ is exact and $I_2^\bullet$ starts at r, and $0 \to \omega_X^0 \to I_2^r \to I_2^{r+1}$ by definition of ω_X^0.

Then $\mathscr{E}xt^r(\mathcal{F},\omega_{\omega^{\mathbb{P}^N}}) = h^r(\operatorname{Hom}_X(\mathcal{F},\mathscr{I}^\bullet)) = h^r(\operatorname{Hom}_X(\mathcal{F},I_2^\bullet)) = \operatorname{Hom}_X(\mathcal{F},\omega_X^0)$. □

Theorem 4.6. *Let X be projective over k. Then there is a morphism $\operatorname{Ext}^i(\mathcal{F},\omega_X^0) \xrightarrow{\theta^i} H^{n-i}(X,\mathcal{F})^*$. The following are equivalent: (1) X is Cohen–Macaulay (resp. S_j) and equidimensional, (2) for any $\mathcal{F}$ locally free, $H^i(X,\mathcal{F}(-q)) = 0$ for q large, all $i > 0$, (resp. for $0 < i < j$) (3) θ^i is an isomorphism for all $i > 0$ (resp. $i \le j$).*

Proof. The left hand side is a universal delta functor as all $\mathcal{F}$ are quotients of $\bigoplus \mathcal{O}(-q)$. Then $\operatorname{Ext}^i(\mathcal{O}(-q), \omega_X^0) = H^i(\omega_X(q)) = 0$. So the θ^i exist.

(1) $\implies$ (2): Let $A = \mathcal{O}_{\mathbb{P}^N,x}$, an N-dimensional regular ring for a closed point $x \in X$. Then $\operatorname{Ext}^j(A, \mathcal{O}_{X,x}) = 0$ if $j > N-n$ since $\operatorname{depth}(\mathcal{O}_{X,x}) = n$. But then $\mathscr{E}xt^j_{\mathbb{P}^N}(\mathcal{F}, \cdot) = 0$ for $j > N-n$ locally free. Then $H^i(X, \mathcal{F}(-q))^* \simeq \operatorname{Ext}^{N-i}_{\mathbb{P}^N}(\mathcal{F}(-q), \omega_{\mathbb{P}^N})$. For q large, this is $\Gamma(\mathscr{E}xt^{N-i}(\mathcal{F}, \omega_{\mathbb{P}^N}(q)))$, and this is 0 for $N-i > N-n$.

(2) $\implies$ (1): the same argument above backwards for $\mathcal{F} = \mathcal{O}_X$ gives vanishing of Ext, which means it is Cohen–Macaulay.

(2) $\implies$ (3): The same argument we used to prove the left side is universal proves that the right side is universal.

(3) $\implies$ (2): Then $H^i(X, \mathcal{F}(-q)) \simeq \operatorname{Ext}^{n-i}(\mathcal{F}(-q), \omega_X^0)^*$ which is $H^{n-i}(X, \mathcal{F}^* \otimes \omega_X^0(q))$ since $\mathcal{F}$ is locally free, and this is 0 for q sufficiently large. ☐

Corollary 4.7. *If X is normal projective of dimension ≥ 2, then for a locally sheaf $\mathcal{F}$, we have $H^1(X, \mathcal{F}(-q)) = 0$ for q large.*

Corollary 4.8. *Let X be normal projective variety over algebraically closed k, and H an ample effective divisor on X. Then H is connected.*

Proof. We may assume H is very ample. Then from $0 \to \mathcal{O}(-qH) \to \mathcal{O}_X \to \mathcal{O}_{qH} \to 0$ we have for q large that $H^1(X, \mathcal{O}(-qH)) = 0$, and since $H^0(X, \mathcal{O}_X) = k$, we also have $H^0(nH, \mathcal{O}_{nH}) = k$, so H is connected. ☐

Theorem 4.9. *Let $\mathcal{F}$ be a locally free sheaf on a Cohen–Macaulay X. Then*

$$H^i(X, \mathcal{F}) \simeq H^{n-1}(\mathcal{F}^* \otimes \omega_X^0))^*.$$

Next we want to see what ω_X^0 is for the smooth case.

Definition 4.10. For A a ring, and $f_1, \ldots, f_n \in A$, we define the Koszul complex K by K_1 being a free A-module with basis $e_1, \ldots, e_r$, and let $K_j = \bigwedge^j K_1$, and differential given by

$$d(e_{i_1} \wedge \cdots \wedge d_{i_p}) = \sum (-1)^{j-1} f_{i_j} e_{i_1} \wedge \cdots \widehat{e_{i_j}} \cdots \wedge e_{i_p}.$$

We denote this by $K_\bullet(f_1, \ldots, f_r; A)$.

Theorem 4.11. *If $f_1, \ldots, f_r$ is a regular sequence, then $K_\bullet(f_1, \ldots, f_r; A) \to A/(f_1, \ldots, f_r) \to 0$ is exact.*

5. 03/03/2020

5.1. **Flat morphisms.** We proved last time:

Proposition 5.1. *$f\colon X \to Y$ flat between schemes of finite type over k, then* $\dim_x X - \dim_y Y = \dim_x(X_y)$.

Corollary 5.2. *Let $f\colon X \to Y$ as above and Y irreducible. Then the following are equivalent:*

(1) every irreducible component of X has dimension $\dim Y + n$,
(2) for every $y \in Y$, every irreducible component of X_y has dimension n.

Proof. (1) $\implies$ (2): Let Z be a irreducible component of X_y. Choose a closed point x on Z but not on any other component. Then $\dim_x X - \dim_y Y = \dim_x X_y$, and using $\dim_x X = \dim X - \dim \overline{\{x\}}$, and since we have $\dim \overline{\{x\}} = \dim \overline{\{y\}}$ as x is closed on X_y. Then $n = \dim X - \dim Y = \dim_x X_y$.

(2) $\implies$ (1): Let Z be a component of X, and $x \in Z$ a closed point (note y is then also a closed point). Then $\dim_x X = \dim Z$. Then $\dim_x X - \dim_y Y = \dim_x X_y = n$, and so again $\dim Z - \dim Y = n$. □

Definition 5.3. A point $x \in X$ is an associate point of X if $\mathfrak{m}_x$ is an associated prime of $0 \in \mathcal{O}_{X,x}$ (same as $\mathfrak{m}_x$ are all zero divisors).

Proposition 5.4. *If $f\colon X \to Y$ with Y integral regular of dimension* 1, *then f is flat if and only if every associated point is mapped to the generic point of Y. In particular, if X is reduced then X is flat if and only if every irreducible component of X dominates Y.*

Proof. If suffices to consider $Y = \operatorname{Spec} \mathcal{O}_{Y,y}$. Let $t \in \mathfrak{m}_y \setminus \mathfrak{m}_y^2$. If f is flat, and $f(x) = y$, then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat. So $0 \to \mathcal{O}_{X,x} \xrightarrow{f^\# t} \mathcal{O}_{X,x}$, so $f^\# t \in \mathfrak{m}_x$ is not a zero divisor, so x is not an associated point of X.

Conversely, we need to prove $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat. If Y is generic, then this is trivial. So let y be a closed point. Then $\mathcal{O}_{Y,y}$ is a DVR, so we need to show $\mathcal{O}_{X,x}$ is torsion free $\mathcal{O}_{Y,y}$-module. If not, then $f^\# t$ is a zero divisor in $\mathfrak{m}_x$, so it is in an associated prime of 0, so this would determine an associated point x' which would map to y, which is a contradiction. □

Example 5.5. Let C be a nodal curve and $\tilde{C} \to C$ its normalization. It is not flat, since $\mathcal{O}_C \to f_*\mathcal{O}_{\tilde{C}}$ is not flat, as then $f_*\mathcal{O}_{\tilde{C}}$ would be locally free since it is coherent. It has dimension 1, but at the double point we need two generators.

Example 5.6. $X \to \mathbb{A}^2$ the blow up at the origin. It is not flat since the dimension of the fibre is greater in the blow up point.

Proposition 5.7. *Let Y be regular of dimension 1 and $p \in Y$ closed. If $X \subseteq_{closed} \mathbb{P}^n_{Y-p} \to Y - p$ is flat, then there exist a unique $\overline{X} \subseteq \mathbb{P}^n_Y$ extending X with $\overline{X} \to Y$ flat.*

Proof. Let $\overline{X}$ be the scheme theoretic closure of $X \subseteq \mathbb{P}^n_Y$. Then the associated points are the same as X, so they all dominate Y, and hence $\overline{X} \to Y$ is flat. □

Example 5.8. Let $X = \operatorname{Spec} k[x, y, t]/(ty - x^2)$ and $Y = \operatorname{Spec} k[t]$. It is flat since the schemes are integral and it is dominant. For $t \neq 0$ we have parabolas, and for $t = 0$ we have a double line, so flat families do not preserve reducedness. Taking $X = \operatorname{Spec} k[x, y, t]/(t = xy)$ we see it do not preserve irreducibility.

Example 5.9. Consider the projection $\varphi\colon \mathbb{P}^{n+1} - (1, 0, \ldots, 0) \to \mathbb{P}^n$. We have $\varphi = \varphi_0$ if $\varphi_a(x_0, \ldots, x_n) = (ax_0, \ldots, x_n)$. If X does not contain $(1, 0, \ldots, 0)$, This gives a family $X \times (\mathbb{A}^1 - 0) \simeq \mathscr{X} \to \mathbb{A}^1 - 0$, and then it extends to a family over $\mathbb{A}^1$.

As an example, consider the twisted curve $\mathbb{P}^1 \to \mathbb{P}^3$. Then the X_0 in this case will be set theoretically a nodal curve, but will have an embedded point "remembering" the projection direction.

Example 5.10. (Flat) family of (Cartier) divisors. Consider $X \to T$ flat for T regular of dimension 1, and a Cartier divisor $D \subseteq X$. We want D to give divisors on each fibre. Let $D = \{(\operatorname{Spec} A, f)\}$, and for $t \in T$ we have $\overline{f} \in A/t$, and it defined a Cartier divisor if $\overline{f}$ are non-zero. This is the same as D being flat over T.

Theorem 5.11. *Let T be integral Noetherian, and $X \subseteq \mathbb{P}^N_T$. Then X is flat over T if and only if the Hilbert polynomials are constant on each fibre.*

Proof. Take $\mathcal{F} = i_* \mathcal{O}_X$, and consider the case $T = \operatorname{Spec} A$ for a local ring A. For any coherent sheaf $\mathcal{F}$, we prove that the following are equivalent: (1) $\mathcal{F}$ is flat over T, (2) $H^0(\mathcal{F}(m))$ is locally free A module for m large, (3) Hilbert polynomials of $\mathcal{F}_t$ are the same.

The only place we need integrality is for (3) $\Longrightarrow$ (1),(2). □

6. 10/03/2020

Last time we discussed smooth morphisms, and that are preserved under base change.

Definition 6.1. $f\colon X \to Y$ is smooth if (i) is flat, (ii) $\dim X' = \dim Y' + n$ for all irreducible components, (iii) $\dim_{k(X)}(\Omega_{X/Y} \otimes k(x)) = n$ for all points x.

Proposition 6.2. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth of relative dimension m, n, so is the composition, of relative dimension $m + n$.*

Proof. (i) and (ii) are easy. For (iii), we use

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

For $x \in X$, let z be the image. Then $f^*\Omega_{Y/Z} \otimes k(x) \to \Omega_{X/Z} \otimes k(x) \to \Omega_{X/Y} \otimes k(x) \to 0$. This implies that $\Omega_{X/Z} \otimes k(x)$ is of rank at most $n+m$. If X' is a irreducible component containing x, then we know that $\Omega_{X/Z} \to \Omega_{X'/Z} \to 0$, so $\Omega_{X'/Z} \otimes k(x)$ has rank $\leq n+m$ and if η is the generic point of Z', we have $\text{rank}\Omega_{X'(\eta)/\eta} \geq m+n$. This implies that $\Omega_{X'/Z}$ is a locally free module of rank $n+m$. □

Corollary 6.3. *Smoothness is preserved under product.*

Proof. Follows from base change and composition. □

Theorem 6.4. *If $f\colon X \to Y$ is a morphism of schemes of finite type over a field k, then f is smooth of relative dimension n if and only if: (1) f is flat, (2) for any $y \in Y$, $X_{\overline{y}} = X \times_{k(y)} \overline{k(y)}$ is equidimensional regular of dimension n.*

Proof. If f is smooth, it is flat and $X_{\overline{y}}$ is smooth on $\overline{y}$. This means $\Omega_{X_{\overline{y}}}$ is a locally free sheaf of rank n, and so $X_{\overline{y}}$ is regular.

Conversely, $X_{\overline{y}}$ being regular implies that $\Omega_{X_{\overline{y}}}$ is a locally free sheaf of rank n. So $\Omega_{X/Y} \otimes \overline{k(y)} = \Omega_{X_{\overline{y}}}$ is locally free of rank n. This implies $\Omega_{X/Y}$ is locally free, since it is coherent, by doing the proof above again. □

For a morphism $f\colon X \to Y$, we have $T_f\colon T_x \to T_{f(x)} \otimes_{k(y)} k(x)$.

Proposition 6.5. *Let $f\colon X \to Y$ a morphism of nonsingular varieties over $k = \overline{k}$. Let $n = \dim X - \dim Y$. Then the following are equivalent: (1) f is smooth of relative dimension n, (2) $\Omega_{X/Y}$ is locally free of rank n, (3) for all $x \in X$, T_f is surjective.*

Proof. (1) $\Longrightarrow$ (2) is trivial, since X, Y are integral.

(2) $\implies$ (3): We have $f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$. Since X, Y are nonsingular, $\Omega_{X/k}$ and $\Omega_{Y/k}$ are locally free, and since by assumption $\Omega_{X/Y}$ is also locally free, and since the ranks add up, this is in fact exact on the left. Taking the dual after tensoring with $k(x)$, we obtain (3).

(3) $\implies$ (1): We know $\mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{k(y)} k(x) \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is injective. Choose $y_1, \ldots, y_r$ a regular sequence where $\dim_y Y = r$, and then the image of them gives a regular sequence on X (since X is regular). Then $\mathcal{O}_{X,x}/(y_1, \ldots, y_r) \to \mathcal{O}_{Y,y}/(y_1, \ldots, y_r) = k(y)$. This is flat, and then using that $M/tM \to A/tA$ flat $\implies$ $M \to A$ flat for t not unit in A, not zerodivisor of A, M, we conclude that f is flat. Now (3) implies the exact sequence $0 \to f^*\Omega_{Y/k} \otimes k(x) \to \Omega_{X/k} \otimes k(x) \to \Omega_{X/Y} \otimes k(x) \to 0$, and since X, Y are nonsingular we conclude that f is smooth as before. □

6.1. **Smoothness on characteristic zero.**

Lemma 6.6. *If $X \to Y$ is dominant morphism of integral schemes of finite type over k of characteristic 0, then there is $U \subseteq X$ open nonempty with $U \to Y$ smooth.*

Proof. Since the singular points on X, Y are closed, we may assume without loss of generality that X, Y are nonsingular. Since we are in characteristic 0, $K(Y) \to K(X)$ is a separable extension. So $\Omega_{X/Y}$ is a free module of rank $\dim X - \dim Y = n$ at the generic point. So it is locally free at an open set of the generic point. By the proposition above, this means f is smooth. □

Remark 6.7. Having such an open set is equivalent to $K(Y) \to K(X)$ being separable. So if $X = Y = \mathbb{P}^1_k$ with f the Frobenius morphism, this is not separable, so f is not smooth in any open set.

Proposition 6.8. *Let $f\colon C \to Y$ a morphism of finite type schemes over k of characteristic 0. Consider $X_r = \{$closed points with $\operatorname{rank}(T_{f,x}) \le r\}$. Then $\dim \overline{f(X_r)} \le r$.*

Proof. Let Y' be an irreducible component of $\overline{f(X_r)}$, and X' an irreducible component of $\overline{X_r}$ which dominates Y'. Give X', Y' the reduced structure, and then since they are integral, we can find $f'\colon U' \subseteq X'$ with $U' \to Y'$ smooth. Then

$$\begin{array}{ccc} T_{x,U'} & \hookrightarrow & T_{x,X} \\ \downarrow\!\!\downarrow {\scriptstyle T_{f',x}} & & \downarrow {\scriptstyle T_{f,x}} \\ T_{y,Y'} & \hookrightarrow & T_{y,Y} \end{array}$$

where the surjectivity is since f' is smooth. Hence $\dim T_{y,Y'} \le r$. Hence $\dim Y' \le r$. □

Corollary 6.9. *Let $f\colon X \to Y$ a morphism between varieties over $k = \overline{k}$ of characteristic 0. Then there exist a nonempty open $V \subseteq Y$ such that $f^{-1}(V) \to V$ is smooth.*

Proof. We may assume Y is nonsingular. Let $r = \dim Y$. Then $\dim f(X_{r-1}) \leq r - 1$. So there is $V \subseteq Y$ with $f^{-1}(U) \cap X_{r-1} = \emptyset$. Then since Y is nonsingular, this implies the $T_{f,x}$ are surjective, and hence $f^{-1}(V) \to V$ is smooth. □

Now let $k = \overline{k}$ of characteristic 0.

Definition 6.10. A homogeneous space is a variety X and a group variety G with an action $\theta\colon G \times X \to X$ which is transitive.

Theorem 6.11. *If X is a homogeneous space, $f\colon Y \to X$ and $g\colon Z \to X$ for nonsingular Y, Z. Then there is $U \subseteq G$ of dimension $\dim Y + \dim Z - \dim X$ such that for any $\sigma \in U(k)$ $Y^\sigma \times_X Z$ is smooth (where Y^σ denotes $Y \xrightarrow{\sigma} Y \to X$).*

Proof. Consider $h\colon G \times Y \to G \times X \to X$. There is an open set $V \subseteq X$ such that $h^{-1}(V) \to V$ is smooth. Then $h^{-1}(V^\sigma) \to V^\sigma$ is smooth. Since X is homogeneous space, this implies that h is smooth. Then $(G \times Y) \times_X Z \to Z$ is smooth. Since Z is smooth, we have that $G \times Y \times_X Z$ is smooth over k. Now projecting this to G, and since G is nonsingular, there is $U \subseteq G$ such that $U \times Y \times_X Z \to U$ is smooth. Hence for every $\sigma \in U$, we get the smooth fiber $Y^\sigma \times_X Z$. Its dimension is $\dim(G \times Y \times_X Z) - \dim G = \dim Y + \dim Z - \dim X$. □

Corollary 6.12. *If X is a nonsingular projective variety has a base point free linear system b, then there is an open set U such that any element in U is smooth.*

Proof. Consider the morphism $X \to \mathbb{P}^n$ determined by the linear system. Since $\mathbb{P}^n$ is homogeneous under $G = PGL(n+1)$, we apply the theorem for $g\colon H \to \mathbb{P}^n$ a hyperplane. So for almost all σ, we have $X \times_{\mathbb{P}^n} H^\sigma = f^{-1}(H^\sigma)$ is smooth. □

7. 12/03/2020

7.1. **Formal schemes.** Mittag-Lefler condition: we have $\varprojlim^1 A_n = 0$ if for all n we have that $\operatorname{Im}(A_m \to A_n)$ is stable for $m \gg 0$. Also recall $\varprojlim^2 A_n = 0$ since it is indexed is by integers.

Theorem 7.1. *Let A be Noetherian, I an ideal. If M_n is an inverse system of finitely generated A/I^n-modules with surjective transitions and $0 \to I^n M_m \to M_m \to M_n \to 0$ for $m > n$, then $\hat{M}$ is finitely generated and $\hat{M}/I^n\hat{M} = M_n$.*

Definition 7.2. For a Noetherian scheme X and Y a closed subscheme, we define the formal completion along Y to be the ringed space $(\hat{X}, \mathcal{O}_{\hat{X}})$. The topological space is $\hat{X} = Y$, but the structure sheaf is $\mathcal{O}_{\hat{X}} = \varprojlim \mathcal{O}_X/\mathscr{I}_Y^n$. For a coherent sheaf $\mathcal{F}$ on X, we define $\hat{F} = \varprojlim \mathcal{F}/\mathscr{I}_Y^n\mathcal{F}$ a sheaf on $\hat{X}$.

Definition 7.3. A Noetherian formal scheme is a locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ that has a finite cover $\{\mathfrak{U}_i\}$ such that each restriction is isomorphic as a locally ringed space to a completion of a Noetherian scheme. A coherent sheaf on $\mathfrak{X}$ is a sheaf such that on each $\mathfrak{U}_i$ it is induces by a coherent sheaf.

Definition 7.4. A Noetherian affine formal scheme is the completion of an affine Noetherian scheme. We denote M^Δ the coherent sheaf associated to a finitely generated module.

Proposition 7.5. *Consider $X = \operatorname{Spec} A$ and let $Y = V(I)$ and $\mathfrak{X} = \hat{X}$. Then $\mathfrak{I} = I^\Delta$ is a sheaf of ideals in $\mathcal{O}_{\mathfrak{X}}$, and $\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^n \simeq (A/I^n)^\sim$. For M finitely generated, $M^\Delta = M^\sim \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathfrak{X}}$, and $M \mapsto M^\Delta$ is exact.*

Definition 7.6. $\mathfrak{I} \subseteq \mathcal{O}_{\mathfrak{X}}$ is an ideal of definition if $\operatorname{Supp}(\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}) = \mathfrak{X}$ and if $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I})$ is a Noetherian scheme.

Proposition 7.7. *If I_1, I_2 are two ideals of definition, then a power of one contains the other. Moreover, there is a unique largest ideal of definition, and it is such that $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I})$ is reduced. Also, if I is an ideal of definition, so is any power.*

Definition 7.8. Given $\varphi\colon (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/I_1)$, we only need to prove $\varphi(I_2)$ is nilpotent. Since we are in the Noetherian case, we need to prove that $\varphi(I_2)$ is contained in all maximal ideals of $p \in \mathfrak{X}$. This is true since the support of I_2 is full.

For the reduced ideal of definition, we note that it is unique if exist, since the proof above shows it is maximal. But then this becomes a local question, and so we may assume we have an affine formal scheme, and this is easy.

Theorem 7.9. *If A is Noetherian and I-adically complete, then $M \mapsto M^\Delta$ is an equivalence of categories between finitely generated modules and coherent sheaves. The inverse is $\Gamma(\hat{X}, \mathcal{F})$.*

Corollary 7.10. *Coherent sheaves on formal schemes form an abelian category.*

8. 31/03/2020

8.1. **Formal function theorem.** Consider $f\colon X \to Y$ a projective morphism with Y Noetherian. For $T \to Y$ with T affine, and for $\mathcal{F}$ on X, we have a morphism $R^i f_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_T \to H^i(X \times_Y T, \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_T)$.

Theorem 8.1 (Grothendieck formal function theorem)**.** *Pick $y \in Y$, and consider $T_n = \mathcal{O}_X/\mathfrak{m}_y^n$. Let $\mathcal{F}$ be a coherent sheaf. Then the morphism*

$$R^i f_* \mathcal{F} \otimes \widehat{\mathcal{O}_{Y,y}} \to \varprojlim_n H^i(X_n, \mathcal{F}_n)$$

is an isomorphism.

Proof. We have $X \hookrightarrow \mathbb{P}^N_Y$, then we may consider the pushforward of $\mathcal{F}$, and so assume $X = \mathbb{P}^N_Y$. We may also assume $Y = \operatorname{Spec} A$ is affine.

If $\mathcal{F} = \mathcal{O}(n)$, we can compute both sides and conclude that the theorem holds.

Since $\mathcal{F}$ is coherent, there is a short exact sequence $0 \to \mathcal{R} \to \mathcal{G} \to \mathcal{F} \to 0$ where $\mathcal{G} = \bigoplus \mathcal{O}(n)$. Tensoring with $A/\mathfrak{m}^n$, we have $0 \to \mathcal{E}_n \to \mathcal{G}_n \to \mathcal{F}_n \to 0$ for some $\mathcal{J}_n$. Then we also have an exact sequence $0 \to \mathcal{J}_n \to \mathcal{R}_n \to \mathcal{E}_n \to 0$.

Then since completion is flat, we have

$$R^i f_* \mathcal{R} \to R^i f_* \mathcal{G} \to R^i f_* \mathcal{F} \to R^{i+1} f_* \mathcal{R} \to R^{i+1} f_* \mathcal{G}$$

and also after completion. Similarly, taking cohomology we get

$$H^i(\mathcal{E}_n) \to H^i(\mathcal{G}_n) \to H^i(\mathcal{F}_n) \to H^{i+1}(\mathcal{E}_{n+1}) \to H^{i+1}(\mathcal{G}_{n+1})$$

and since they are finite modules, they satisfy Mittag–Leffler, so we can take the inverse limit.

Now we have isomorphisms in these two exact sequences for $\mathcal{G}$:

$$\begin{array}{ccccccccc}
\hat{R}^i f_* \mathcal{R} & \longrightarrow & \hat{R}^i f_* \mathcal{G} & \longrightarrow & \hat{R}^i f_* \mathcal{F} & \longrightarrow & \hat{R}^{i+1} f_* \mathcal{R} & \longrightarrow & \hat{R}^{i+1} f_* \mathcal{G} \\
\downarrow & & \Big\downarrow \sim & & \Big\downarrow & & \downarrow & & \Big\downarrow \sim \\
\varprojlim H^i(\mathcal{R}_n) & & & & & & \varprojlim H^{i+1}(\mathcal{R}_{n+1}) & & \\
\downarrow & & & & & & \downarrow & & \\
\varprojlim H^i(\mathcal{E}_n) & \longrightarrow & \varprojlim H^i(\mathcal{G}_n) & \longrightarrow & \varprojlim H^i(\mathcal{F}_n) & \longrightarrow & \varprojlim H^{i+1}(\mathcal{E}_n) & \longrightarrow & \varprojlim H^{i+1}(\mathcal{G}_n)
\end{array}$$

Claim: $\varprojlim H^i(\mathcal{R}_n) \simeq \varprojlim H^i(\mathcal{E}_n)$.

Then we do decreasing induction. For $i \geq N+1$, the theorem is true since both are 0. Then together with a diagram chasing we conclude the induction. □

Proof of the claim. We will prove that for $n' \gg n$, we have $\mathcal{J}_{n'} \to \mathcal{J}_n$ is the zero morphism.

We may prove this locally. We have $\mathcal{J}_n = \ker(\mathcal{R}/\mathfrak{m}^n\mathcal{R} \to \mathcal{G}/\mathfrak{m}^n\mathcal{G})$. So $\mathcal{J}_n = (\mathcal{R} \cap \mathfrak{m}^n\mathcal{G})/\mathfrak{m}^n\mathcal{R}$. By Krull's theorem, $R \cap \mathfrak{m}^{n'}\mathcal{E} \subseteq \mathfrak{m}^n\mathcal{R}$ for $n' \gg$. □

Remark 8.2. One can relax f to be proper by using Chow's lemma.

Remark 8.3. One can develop the cohomology theory for formal schemes. Then both sides will be $H^i(\mathfrak{X}, \mathfrak{F})$.

Corollary 8.4. *Let $f\colon X \to Y$ a projetive morphism with Y Noetherian. Let $r = \max\{\dim f^{-1}(y)\colon y \in Y\}$. Then $R^i f_*\mathcal{F} = 0$ for $\mathcal{F}$ coherent and $i > r$.*

Corollary 8.5. *Assume furthermore that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then $f^{-1}(y)$ is connected for any $y \in Y$.*

Proof. Consider $f^{-1}(y) = X' \sqcup X''$. Then $H^0(X_n, \mathcal{O}_{X_n}) = H^0(X'_n, \mathcal{O}_{X'_n}) \oplus H^0(X''_n, \mathcal{O}_{X''_n})$, and by the theorem we get $\hat{\mathcal{O}_Y} = \varprojlim H^0(X'_n, \mathcal{O}_{X'_n}) \oplus \varprojlim H^0(X''_n, \mathcal{O}_{X''_n})$, which is not possible for local rings. □

Corollary 8.6 (Zariski's Main theorem)**.** *$X \to Y$ birational projective with Y Noetherian for X, Y integral. Assume also Y is normal. Then for any $y \in Y$, $f^{-1}(y)$ is connected.*

Proof. $f_*\mathcal{O}_X$ is a finite $\mathcal{O}_Y$-module, and since Y is normal and X, Y are birational, we have $f_*\mathcal{O}_X = \mathcal{O}_Y$. □

Theorem 8.7 (Stein factorization)**.** *For $f\colon X \to Y$ projective with Y Noetherian, there is Z such that $f\colon X \xrightarrow{f'} Z \xrightarrow{g} Y$ and g is finite, f' has connected fibres.*

Proof. Let $Z = \operatorname{Spec}_{\mathcal{O}_Y}(f_*\mathcal{O}_X)$. Then g is finite as $f_*\mathcal{O}_X$ is finite $\mathcal{O}_Y$-module, and we have $f'_*\mathcal{O}_X = \mathcal{O}_Z$. □

8.2. **Semicontinuity.** The setting is $f\colon X \to Y$ projective. (Again can do for proper). F is a coherent sheaf on X. We want to compare $R^i f_*\mathcal{F} \otimes k(y)$ and $H^i(X, \mathcal{F}_y)$. We also assume $\mathcal{F}$ is flat over Y.

Assume without loss of generality that $Y = \operatorname{Spec} A$.

We look the more general $T^i(M) := H^i(X, \mathcal{F} \otimes_A M)$ for an A-module M. Then T is a δ-functor from flatness.

Proposition 8.8. *There exist a complex $L^\bullet$ for L^i finite free A-modules such that $T^i(M) \simeq h^i(L^\bullet \otimes_A M)$.*

Proof. First, take a Cech covering of X and form the Cech complex $C^\bullet = C^\bullet(\mathscr{U}, \mathcal{F})$. Then $C^\bullet$ are flat over A. Then $C^\bullet(\mathscr{U}, \mathcal{F} \otimes_A M) = C^\bullet \otimes_A M$. Then $T^\bullet(M) = h^i(C^\bullet \otimes_A M)$.

However, $C^\bullet$ are not finite free A-modules.

We will prove: for any $C^\bullet$ such that $h^i(C^\bullet)$ are finite A-modules, there is $L^\bullet$ finite free with a morphism $g\colon L^\bullet \to C^\bullet)$ such that $h^i(C^\bullet) \simeq h^i(L^\bullet)$. Moreover, if $C^\bullet$ are flat, then $h^i(L^\bullet \otimes_A M) = h^i(C^\bullet \otimes_A M)$.

We construct $L^\bullet$ by decreasing induction. For $i \gg 0$, we have $h^i(C^\bullet) = 0$, so take $L^i = 0$. Assume L^i is constructed for $\geq i+1$ such that $L^\bullet \to C^\bullet$ induce an isomorphism for h^j with $j \geq i+2$, and such that $Z^{i+1}(L^\bullet) \to h^{i+1}(C^\bullet)$ is surjective. $h^i(C^\bullet)$ is generated by $x_1, \dots, x_n$, and $(g^{i+1})^{-1}(B^{i+1}(C^\bullet)) \subseteq L^{i+1}$ is finitely generated, say $y_{r+1}, \dots, y_s$. Let $L^i := \bigoplus_{j=1}^s e_j$, and let $e_j \mapsto y_j$ if $j > r$ and let $g^i(e_j)$ be a lift of x_j for $j \leq r$, and be h_j for $j > r$ such that $h_j \mapsto f(e_j)$ in C^{i+1}.

When $C^\bullet$ is flat, we need to prove $h^i(L^\bullet \otimes M) = h^i(C^\bullet \otimes M)$. For $i \gg 0$, both are 0. If M is a free A-module, this is true. Since it commutes with direct limits, we can assume M is finitely generated. Taking $0 \to E \to R \to M \to 0$ for R finite free, we can tensor with $L^\bullet$ and $C^\bullet$ and take cohomology. Here we are using that $C^\bullet$ and $L^\bullet$ are flat. Then by a diagram chasing we perform descending induction. $\square$

9. 02/04/2020

Question: in what cases do we have $T^i(k(y)) = T^i(A) \otimes k(y)$?

Proposition 9.1. *The following are equivalent: (i) T^i is left exact, (ii) $W^i := \operatorname{coker}(L^{i-1} \to L^i)$ is a projective A-module, (iii) there is a finitely generated A-module Q such that $T^i(M) = \operatorname{Hom}(Q, M)$.*

Proof. Since $W^i(L^\bullet \otimes M) = W^i \otimes M$, we have $T^i(M) = \ker(W^i \otimes M \to L^\bullet \otimes M)$. To see that (i) and (ii) are equivalent, for $M' \hookrightarrow M$, consider

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^i(M') & \longrightarrow & W^i \otimes M' & \longrightarrow & L^i \otimes M' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T^i(M) & \longrightarrow & W^i \otimes M & \longrightarrow & L^i \otimes M \end{array}$$

and so the first arrow is injective if and only if the second is, and the second is injective if and only if W^i are flat. Since it is finitely generated, this is the same as projective.

For (iii), it implies (i) trivially. Assuming (ii), the duals of $L^\bullet$ and $W^\bullet$ are projective. So let $Q = \operatorname{coker}(\check{L}^{i+1} \to \check{W}^i)$, and then we have

$$0 \to \operatorname{Hom}(Q, M) \to \operatorname{Hom}(\check{W}^i, M) \to \operatorname{Hom}(\check{L}^{i+1}, M)$$

and using that they are projective, the last two terms are $M \otimes W^i \to M \otimes L^{i+1}$. □

Remark 9.2. In (iii), such Q is necessarily unique.

Proposition 9.3. *There is a morphism $\varphi \colon T^i(A) \otimes M \to T^i(A \otimes M)$, and the following are equivalent: (i) T^i is right exact, (ii) φ is an isomorphism for all M, (iii) φ is surjective for all M.*

Proof. φ is given by $M = \operatorname{Hom}(A, M) \to \operatorname{Hom}(T^i(A), T^i(M))$, which is adjoint to $M \otimes T^i(A) \to T^i(M)$.

For any M, we may assume it is finitely generated, since everything commutes with direct limits. So consider $A^s \to A^r \to M \to 0$. Then we have

$$\begin{array}{ccccccc} T^i(A) \otimes A^s & \longrightarrow & T^i(A) \otimes A^r & \longrightarrow & T^i(A) \otimes M & \longrightarrow & 0 \\ \downarrow\wr & & \downarrow\wr & & \downarrow & & \\ T^i(A^s) & \longrightarrow & T^i(A^r) & \longrightarrow & T^i(M) & & \end{array}$$

And then it is clear that (i) $\implies$ (ii).

So it remains to do (iii) $\implies$ (i). For $M \to M' \to 0$, we have the diagram

$$\begin{array}{ccc} T^i(A) \otimes M & \twoheadrightarrow & T^i(A) \otimes M' \\ \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow \\ T^i(M) & \longrightarrow & T^i(M') \end{array}$$

and so the bottom map is also surjective. □

Theorem 9.4. *T^i is exact if and only if T^i is right exact and $T^i(A)$ is projective.*

Proof. T^i being right exact implies $T^i(M) \simeq T^i(A) \otimes M$, and being left exact then means $T^i(A)$ is flat, and hence projective. □

Now we consider the local situation by base changing to A_y.

Proposition 9.5. *If T^i is left/right exact at some point $y \in Y$, then the same holds for an open neighborhood of $y \in Y$.*

Proof. T^i is left exact at y if and only if W^i is locally free at y, and since W^i is finitely generated A-module, this implies it is locally free in a neighborhood.

T^i is right exact if and only if T^{i+1} is left exact, so the same follows. □

Definition 9.6. Let X be a topological space. $\varphi\colon X \to \mathbb{Z}$ is upper semicontinuous if for each y, there is a neighborhood U such that $\varphi(U) \leq \varphi(y)$. Equivalently, $\varphi^{-1}(n, \infty)$ is closed.

Example 9.7. For X a Noetherian scheme and $\mathcal{F}$ a coherent sheaf, $\varphi(y) = \text{rank}\mathcal{F} \otimes k(y)$ is upper semicontinuous.

Theorem 9.8. *Let $\mathcal{F}$ be a flat sheaf on Y. Then $h^i(y, \mathcal{F}) := \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$ is upper semicontinuous.*

Proof. $h^i(y, \mathcal{F}) = \dim_{k(y)} T^i(k(y))$, and we have

$$0 \to T^i(k(y)) \to W^i \otimes k(y) \to L^{i+1} \otimes k(y) \to W^{i+1} \otimes k(y) \to 0.$$

So $\dim T^i(k(y)) = \dim W^i \otimes k(y) + \dim W^{i+1} \otimes k(y) - \dim L^{i+1} \otimes k(y)$. The two first terms are upper semicontinuous and the third is constant. □

Corollary 9.9 (Gauert)**.** *Let $\mathcal{F}$ be a flat sheaf on Y, and Y integral. Assume $h^i(y, \mathcal{F})$ is constant. Then $R^i f_*(\mathcal{F})$ is locally free and $R^i f_*(\mathcal{F}) \otimes k(y) \simeq H^i(X_y, \mathcal{F}_y)$.*

Proof. By the proof above, both $W^i \otimes k(y)$ and $W^{i+1} \otimes k(y)$ have constant dimension. Since Y is integral and $W^\bullet$ are finite, this implies W^i and W^{i+1} are locally free. So T^i, T^{i+1} are left exact, and so T^i is exact. This implies that $T^i(A) = R^i f_*(\mathcal{F})$ is locally free and $T^i(A) \otimes k(y) \otimes T^i(A \otimes k(y))$. □

Example 9.10. Consider the twisted rational cubic in $\mathbb{P}^3$, with the degeneration to a nodal curve. Then $h^0(C_t, \mathcal{O}_t) = 1$ for $t \neq 0$, but $h^0(C_0, \mathcal{O}_0) = 2$, and $h^1(C_t, \mathcal{O}_t) = 0$ for $t \neq 1$ and $h^1(C_0, \mathcal{O}_0) = 1$.

Example 9.11. Let $R = \mathbb{C}$, and consider $X \to Y$ a projective smooth family. Then $h^i(X_t, \mathcal{O}_t)$ is constant: we know $h^i(X_t, \Omega_t^j)$ is constant by Hodge theory, the betti cohomology (their sum) are constants by topology, and they are all upper semicontinuous, so they must all be constant.

Proposition 9.12. *If $\varphi\colon T^i(A)\otimes k(y)\to T^i(k(y))$ is surjective, then T^i is right exact at y.*

Proof. We may assume $A=A_y$. We need to prove $T^i(A)\otimes M\to T^i(M)$ is surjective for all M.

Assume M is a A-module with finite length over $k(y)$. Then this follows from an induction. Writing $0\to M'\to M\to M''\to 0$ with M'' of length 1, we tensor with $T^i(A)$ and we take T^i, use the induction hypothesis.

Now we use the formal function theorem: assume M is finite, and we know now that $T^i(A)\otimes M/\mathfrak{m}^nM\simeq T^i(M/\mathfrak{m}^nM)$, and from the formal function theorem we have $T^i(A)\otimes\hat{M}\simeq T^i(M)\otimes\hat{A}$. Hence $(T^i(A)\otimes M)\otimes\hat{A}=T^i(M)\otimes\hat{A}$, and since A is local, we have $T^i(A)\otimes M=T^i(M)$. □

Theorem 9.13 (Cohomology and base change)**.** *Let $X\to Y$ projective and $\mathcal{F}$ coherent on X and flat over Y. There is a natural map $\varphi^i(y)\colon R^if_*(\mathcal{F})\otimes k(y)\to H^i(X_y,\mathcal{F}_y)$. If it is surjective, it is an isomorphism, and then it holds for a neighborhood. Assuming $\varphi^i(y)$ is surjective, then the following are equivalent: (i) $\varphi^{i-1}(y)$ is surjective and (ii) $R^if_*(\mathcal{F})$ is locally free.*

Proof. If it is surjective, T^i is right exact at y, so $T^i(M)\simeq T^i(A)\otimes M$, so it iso.

(i) $\Longleftrightarrow$ T^{i-1} right exact, which is T^i left exact, so T^i exact, so $R^if_*(\mathcal{F})$ locally free=(ii). □

10. 07/04/2020

10.1. **Hilbert Schemes.** For a projective X/S, consider the functor $Hilb\colon$ {schemes over S} $\to$ Set by

$$Hilb(T)=\{\text{closed subschemes } Y\subseteq X\times_S T \text{ and } Y\to T \text{ flat.}\}$$

This is a disjoint union based on different Hilbert polynomials, denote $Hilb_f$ when we restrict to the ones with such f.

Theorem 10.1 (Grothendieck)**.** *$Hilb_f$ is represented by a projective scheme Hilb_f.*

Example 10.2. The identity on Hilb_f correspond to $U\subseteq\mathrm{Hilb}_f\times_S X$ flat over Hilb_f is a universal family.

Theorem 10.3 (Universal vanishing theorem)**.** *Let X be projective over k, and $\mathcal{E}$ a locally free sheaf on X. Fix a polynomial f. There is a universal d_0 such that for any $0\to\mathcal{F}\to\mathcal{E}$, with $\chi(\mathcal{F}(n))=f(n)$ (for n large) then for $d\geq d_0$ we have $H^i(X,\mathcal{F}(d))=0$ and $\mathcal{F}(d)$ globally generated.*

Definition 10.4. A coherent sheaf on X is m-regular if $H^i(X,\mathcal{F}(m-i))=0$ for all $i>0$.

Theorem 10.5 (Castenuelvo–Mumford regularity theorem)**.** *If $\mathcal{F}$ is m-regular, then it is also $m+1$-regular and $H^0(X, \mathcal{O}(1)) \times H^0(X, \mathcal{F}(m)) \to H^0(X, \mathcal{F}(m+1))$ is surjective.*

Corollary 10.6. *(i) $H^i(\mathcal{F}(d)) = 0$ for all $d \geq m - i$ for $i > 0$, (ii) $\mathcal{F}(m)$ is globally generated.*

Proof. For (ii), we have $\mathrm{Sym}^d H(X, \mathcal{O}(1)) \times H^0(X, \mathcal{F}(m)) \to H^0(X, \mathcal{F}(m+d))$ is surjective. For $d \gg 0$, $\mathcal{F}(m+d)$ is globally generated, and we can use this to conclude $\mathcal{F}(m)$ is globally generated. □

Proof of regularity theorem. Take H a hyperplane section. For H general, we have $0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0$. To justify why $\mathcal{F}(-1) \to \mathcal{F}$ is injective, work locally.

We now prove have that $\mathcal{F}_H$ is m-regular just from the exact sequence. We do induction on dimension, so that the theorem holds for H. Then $\mathcal{F}_H$ is $(m+1)$-regular. From the same exact sequence, we then conclude that $\mathcal{F}$ is $(m+1)$-regular.

To prove the surjection part, again by induction. □

Proof of universal vanishing. Assume first $X = \mathbb{P}^N$. We may assume without loss of generality that $\mathcal{E} = \bigoplus \mathcal{O}(l_i)$. Again we do induction. Take a good H such that $0 \to \mathcal{F}_H \to \mathcal{E}_H$. Regardless of H, $\mathcal{E}_H$ is a fixed sheaf on $\mathbb{P}^{N-1}$. Now $\chi(\mathcal{F}_H(n)) = \chi(\mathcal{F}(m)) - \chi(\mathcal{F}(m-1))$, so $\mathcal{F}_H$ also has a fixed Hilbert polynomial. So we can apply induction. Then from all the above we get the vanishing for $i \geq 2$. To get it for i, look at the exact sequence, and then use the surjectivity part of the regularity theorem, we can prove by descending induction that $H^0(\mathcal{F}(d)) \to H^0(\mathcal{F}_H(d))$ is surjective. So if $H^1(\mathcal{F}(d-1)) \neq 0$, we would have $\dim H^1(\mathcal{F}(d)) < \dim H^1(\mathcal{F}(d-1))$, and so $H^1(\mathcal{F}(d))$ must be 0 for d large. To prove such large does not depend on the choices, we use the Hilbert polynomial.

For the general situation, embed in $\mathcal{O}_X(l)$, and then consider the sheaf above $\mathcal{F}$ on $\mathbb{P}^n$, and apply the theorem for $\mathbb{P}^N$.

The global generatedness now follows from the vanishing theorem since $\mathcal{F}(d)$ is 0 regular for $d \gg 0$. □

11. 09/04/2020

11.1. **Flatness.**

Theorem 11.1 (Flattening stratification theorem)**.** *Let $X \to S$ projective, S Noetherian, $\mathcal{F}$ a coherent sheaf. There exists a unique stratification S_i such that $S = \bigcup S_i$, each S_i is locally closed subscheme of S, such that $\mathcal{F}|_{S_i}$ is flat, and for every morphism $g\colon T \to S$, then $\tilde{G}^*\mathcal{F}$ over X_T is flat over T if and only if T factors through a finite piece of the stratification.*

Proposition 11.2. *We first prove a weaker version: S is reduced Noetherian, $f\colon X \to S$ a finite type, and $\mathcal{F}$ coherent on X. Then there exist a non-empty open subset $U \subseteq S$ such that $\mathcal{F}|_U$ is flat over U.*

Corollary 11.3. *Applying Noetherian induction on S, there exist a finite stratification V_i of locally closed subsets of S such that $\mathcal{F}$ is flat over V_i.*

Proof of proposition. We may assume that $S = \operatorname{Spec} A$. We may also assume $X = \operatorname{Spec} B$ for B a finite A-algebra. Over the generic points, the sheaf is flat.

$\mathcal{F}$ over B is a finitely generated B-module, and we can find a filtration $M \supset M_1 \supset \cdots \supset M_{n+1} = 0$ with $M_i/M_{i+1} \simeq B/\mathfrak{p}_i$. Now we find an open set $\operatorname{Spec} A_f$ such that for all i $(B/\mathfrak{p}_i)_f$ is flat over A_f. This is an exercise in commutative algebra. □

Now we want some universal vanishing theorem for fibers. From now on, we have the setup of theorem.

Lemma 11.4. *There exist d_0 such that $H^i(X_s, \mathcal{F}_s(d)) = 0$ for any $d \geq d_0$, $i > 0$ and $s \in S$.*

Proof. Applying the corollary above, the problem reduces to the case where $\mathcal{F}$ is flat over S. We can also assume $S = \operatorname{Spec} A$. There is a d_0 such that $H^i(X, \mathcal{F}(d)) = 0$ for $d \geq d_0$ and $i > 0$. So then $0 \to H^0(\mathcal{F}(d)) \to C^0(\mathcal{F}(d)) \to C^1(\mathcal{F}(d)) \to \cdots \to C^k(\mathcal{F}(d)) \to 0$. But as $\mathcal{F}$ is flat, each $C^i(\mathcal{F}(d))$ is also flat. Then tensoring with $k(s)$, we get

$$0 \to H^0(\mathcal{F}(d)) \otimes k(s) \otimes C^0(\mathcal{F}(d)) \otimes k(d) \to C^1(\mathcal{F}(d)) \otimes k(d) \to \cdots \to C^k(\mathcal{F}(d)) \otimes k(s) \to 0.$$

But $C^i(\mathcal{F}(d)) \otimes k(s) = C^i(\mathcal{F}_s(d))$, and so $H^0(\mathcal{F}(d)) \otimes k(s) = H^0(\mathcal{F}_s(d))$ and $H^i(\mathcal{F}_s(d)) = 0$. □

Lemma 11.5. *For each $V_i = \operatorname{Spec} A_i$, when $S = \operatorname{Spec} A$, and A_i finitely generated over A, there is d_i such that $H^0(X_{A_i}, \mathcal{F}_{A_i}(d)) = H^0(X, \mathcal{F}(d)) \otimes A_i$ for $d \geq d_0$. (here we are not assume $\mathcal{F}$ is flat over V_i)*

Proof. For every finitely generated A-module M, we will prove $H^0(X, \mathcal{F}(d) \otimes M) = H^0(X, \mathcal{F}(d)) \otimes M$ for $d \gg 0$ depending on M.

If M is free, this is trivially true. In general $0 \to R \to A^n \to M \to 0$, then $R \otimes \mathcal{F}(d) \to A^n \otimes \mathcal{F}(d) \to M \otimes \mathcal{F}(d) \to 0$, and for d large we still have

$$H^0(R \otimes \mathcal{F}(d)) \to H^0(A^n \otimes \mathcal{F}(d)) \to H^0(M \otimes \mathcal{F}(d)) \to 0,$$

and we also have

$$R \otimes H^0(\mathcal{F}(d)) \to A^n \otimes H^0(\mathcal{F}(d)) \to M \otimes H^0(\mathcal{F}(d)) \to 0$$

and we have a morphsism form the second one to the first. This implies the map we want is surjective for d large. But then the same is true for $R \otimes H^0(\mathcal{F}(d)) \to H^0(R \otimes \mathcal{F}(d))$, which then implies the map we want is an isomorphism.

So now we want to break $V_i \to S$ into flat and a finite morphisms, since the theorem is true for both. If A_i is finitely generated, then $A \to A/\ker(A \to A_i) = B$ is finite, and then break $B \to B[x_1, \dots, x_n] \to A_i$, since the first is flat, and the second finite. □

Proposition 11.6. *For $B \to A$, $\mathcal{F}_B$ is flat over B if and only if $g^* f_* \mathcal{F}(d)$ is locally free for all $d \gg 0$.*

Proof. For $d \gg 0$ we have $g^* f_* \mathcal{F}(d) \simeq f_* \tilde{g}^* \mathcal{F}(d)$ for d large. Then the proposition follows from the result that a sheaf is flat if and only if the pushforward of sufficiently large twists is locally free. □

For $\mathcal{E}$ a coherent sheaf on S, if $\mathcal{E}(s)$ is of rank e, we can find a neighborhood U of e such that

$$\mathcal{O}_U^f \xrightarrow{\psi} \mathcal{O}_U^e \to \mathcal{E} \to 0.$$

Then the locus where $\psi = 0$ is precisely the subscheme $V_e \subseteq U$ where $\mathcal{E}|_{V_e}$ is locally free of rank e: if $T \to U$ is ith $\mathcal{E}_T$ is locally free of rank e, then $T \to V_e \to U$. Then we can patch the V_e together to get a subscheme of S.

We want to find the locus where $g^* f_* \mathcal{F}(d)$ are all locally free for $d \geq d_0$. For each connected component we have the Hilbert polynomial of $f(d_0), f(d_1), \dots, f(d_0 + n)$.

We look at the locus on S where $f_*(\mathcal{F}(d_0 + i))$ has rank $f(d_0 + i)$ for $0 \leq i \leq n$. This gives a subscheme $T \subseteq S$ such that $\chi(\mathcal{F}_t(d_0+i)) = H^0(\mathcal{F}_t(d_0+i)) = f(d_0+i)$. This means that the Hilbert polynomial of $\mathcal{F}_r$ is indeed f. Hence $\mathcal{F}|_{T_{red}}$ is flat (we proved for integral, but reduced suffices). Now we know $f_*(\mathcal{F}(d+i))$ is locally free on T for $0 \leq i \leq n$. Looking at $f_*(\mathcal{F}(d+n+1)), \dots$, we can keep shrinking the reduced structure to make each one work (in the maximal way possible). But the Noetherianess guarantees that this stabilize, and so we can change T to something between T and T_{red} to make all of them locally free.

These T for varying e and Hilbert polynomial prove the stratification theorem.

12. 14/04/2020

Proposition 12.1. *Let $X \to \operatorname{Spec} A$ be projective, $\mathcal{F}$ coherent on X. Then for any finitely generated $\operatorname{Spec} A_i \to \operatorname{Spec} A$, there is d_0 such that $H^0(\mathcal{F}(d)) \otimes A_i = H^0(\mathcal{F}_{A_i}(d))$ for $d \geq d_0$.*

Proof. Let $S = \bigoplus H^0(\mathcal{F}(d))$ with $\tilde{S} \simeq \mathcal{F}$. Then $\tilde{S}_{A_i} = (S \otimes A_i)\tilde{} = \mathcal{F}_{A_i}$, and so the H^0 agree for d large. □

Definition 12.2. The Grassmanian $Gr(m, n)/k$ over a field k for $m < n$ parametrizes m-dimensional subspaces of k^n.

For $m = 1$, $Gr(m, n) = \mathbb{P}^{n-1}$. On any Grassmanian, we have

$$0 \to K \to \mathcal{O}^{\oplus n} \to S \to 0$$

where K is a rank m vector bundle, and S is a rank $n - m$ vector bundle, the universal quotient bundle.

It represents the functor

$$(T \to S) \mapsto \{\mathcal{O}_T^n \to Q \to 0 \text{ for } Q \text{ a locally free of rank } n - m\}$$

For a Noetherian scheme S, a projective S-scheme X and a coherent $\mathcal{E}$ on X, fix a Hilbert polynomial $P(d)$. Consider the functor

$$(T \to S) \mapsto \{\mathcal{E}_T \to Q \to 0 \text{ with } Q \text{ flat}/T \text{ with Hilbert polynomial } P(d)\}$$

where $\mathcal{E}_T$ is on X_T.

Theorem 12.3 (Grothendieck Quotient scheme)**.** *The functor above is representable by a projective scheme* $\operatorname{Quot}(\mathcal{E}, P(d))$.

Example 12.4. For $X = S$, $\mathcal{E} = \mathcal{O}_S^{\oplus n}$ and $P(d) = n - m$, then it is given by the Grassmanian.

Proof of theorem. Since the representable functor is unique, it suffices to consider the case $S = \operatorname{Spec} A$.

First consider the case $X = \mathbb{P}_S^N$ and $\mathcal{E} = \mathcal{O}_X(l)^{\oplus n}$.

Consider $0 \to \mathcal{K} \to \mathcal{E}_T \to Q \to 0$. Then the Hilbert polynomial of $\mathcal{K}$ is also fixed, say $P_1(d)$. By the universal vanishing theorem, there is a d_0 not depending on Q such that $H^i(\mathcal{K}(d)) = 0$ (in fact, $R^i\pi_*\mathcal{K}(d) = 0$) for $i > 0$ and $d \geq d_0$. Then if $\pi\colon X_T \to T$, we have

$$0 \to \pi_*\mathcal{K}(d) \to \pi_*\mathcal{E}_T(d) \to \pi_*Q(d) \to 0.$$

Choose d with $\mathcal{K}$ globally generated also. The middle term is $(\mathcal{O}_S)^{\oplus N}$ for $N = \binom{N+l+d-1}{l+d}$, and $\pi_*Q(d)$ is locally free sheaf on S (as Q is flat) with rank $P_1(d)$.

Then this data induce a morphism $T \to Gr(M := P(d), N)$ (such that $\pi_*Q(d)$ is the pullback of the quotient bundle). This parametrizes the quotient $\mathcal{F} := \pi_*Q(d)$, so we have to cut out a subscheme that correspond to the $\mathcal{F}$ that come from a Q. Look at $\mathcal{K}' := \pi^*\pi_*\mathcal{E}(d) \to \mathcal{E}(d) \to \mathcal{Q}' \to 0$ on $\mathbb{P}^N_T$. Now let $\pi_{Gr}\colon \mathbb{P}^N_{Gr} \to Gr$, then the first morphism is a pullback of the composition $\pi^*_{Gr}\mathcal{K}' \to \pi^*_{Gr}\mathcal{O}^{\oplus n} \to \bigoplus^N_{Gr}(d)$.

Now apply the flat stratification for $\mathbb{P}^N_{Gr} \to Gr$, so that we have a locally closed subscheme $Qu \subseteq Gr$ such that $\mathcal{Q}_{Qu}$ is flat of Hilbert polynomial. Then the map $T \to Gr$ we construct in fact factors through Qu. This is the representing scheme.

We know that Qu is quasi-projective, so it remains to prove it is proper. We need to prove there is a unique flat extension, this is not too hard.

For the general case, one can pushforward fo $\mathbb{P}^N$ after a twist and use the special case. □

13. 16/04/2020

13.1. **Construction of Hilbert scheme.** Consider $X \to S$ and consider the quotients $\mathcal{O}_T \to \mathbb{Q} \to 0$ such that Q is flat over T. The associated Quot scheme is the Hilbert scheme.

13.2. **Deformation Theory.**

Theorem 13.1. *The Zariski tangent space to* $\mathrm{Quot}(\mathcal{O}^n_X, P)$ *at* q *is isomorphic to* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q}\mathcal{Q})$, *where* q *is given by the data* $0 \to \mathcal{K} \to \mathcal{O}^n_X \to \mathcal{Q} \to 0$.

Proof. We may assume that $k = k(q)$. Then if $q\colon \operatorname{Spec} k \to \mathrm{Quot}$, then the tangent space is given by points $\operatorname{Spec} k[\epsilon]/\epsilon^2$ where q factor. Let $A = k[\epsilon]/\epsilon^2$. Then such a map correspond to $0 \to \mathcal{K}_A \to \mathcal{O}^n_A \to \mathcal{Q}_A \to 0$, and when reduced with $\operatorname{Spec} k \to \operatorname{Spec} A$ gives the original exact sequence.

This follows from a problem in the homework. □

Theorem 13.2. *If* $\mathrm{Ext}^1(\mathcal{K}, \mathcal{Q}) = 0$, *then* $\mathrm{Quot}(\mathcal{O}^n_X, P)$ *is nonsingular at* p.

Theorem 13.3. *In any case,* $\dim_q \operatorname{Quot}(\mathbb{P}^n_X, P) \geq \dim \operatorname{Hom}_{\mathcal{O}}(\mathcal{K}, \mathcal{Q}) - \dim \operatorname{Ext}^1_{\mathcal{O}}(\mathcal{K}, \mathcal{Q})$.

To prove it is smooth at q, this is if and only if for any Artinian algebra morphism $B \to A \to 0$ and $\mathcal{O} \to A$ there is a lift to B.

14. 21/04/2020

Sketch of proof of 13.2. To prove Quot is smooth at q, it sufficed to prove that it is formally smooth, that is $\operatorname{Spec} A \to \operatorname{Quot}$ always extends to $\operatorname{Spec} B \to \operatorname{Quot}$ for $B \to A \to 0$ Artinian rings. We may assume $0 \to J \to B \to A$ with $\mathfrak{m} \cdot J = 0$, that is, J is a k-vector space.

Now for a surjection $B \to A$, and since Q_A is flat over A, we have Q_B flat over B if and only if $\operatorname{Tor}^1_B(Q_B, A) = 0$. But this is $\ker(J \otimes kQ_B \to Q_B) = 0$. But $J \otimes_k Q_B = J \cdot Q_B = Q_B \cap J \cdot T^n_B$, so we need to show this is 0.

In short, we do one deformation at a time (this is the $\mathfrak{m} \cdot J = 0$), analyze it locally and then glue with the spectral sequence. □

Sketch of proof of 13.3. There is a regular ring R of $\mathcal{O}_q$ of dimension the tangent space. Letting $0 \to I \to R \to \mathcal{O}_q \to 0$, we prove that $\dim_k I \otimes_R k \leq \dim_k \operatorname{Ext}^1(\mathcal{K}, \mathcal{Q})$. If this is true, then I is generated by at most $\dim_k \operatorname{Ext}^1(\mathcal{K}, \mathcal{Q})$ elements by Nakayama. □

Example 14.1. If $Y \subseteq X$ is a closed subscheme with ideal sheaf I, then if X, Y are smooth projective over k, we have $\operatorname{Hom}(I, \mathcal{O}_Y) = H^0(Y, N_{Y/X}$ and $\operatorname{Ext}^1_{\mathcal{O}_X}(I, \mathcal{O}_Y) \simeq H^1(Y, N_{Y/X})$.

Example 14.2. If X, Y are projective, then $\operatorname{Hom}(X, Y)$ is an open scheme of $\operatorname{Hilb}(Y \times X)$. This is since the normal bundle of $Y \xrightarrow{\text{graph}} Y \times X$ is $g^* T_x$.

Example 14.3. If C is smooth projective, then $\operatorname{Hilb}_n(C) \simeq \operatorname{Sym}^n(C)$.

Example 14.4. This is not true for smooth projective surfaces.

14.1. **Mori's bend and break theorem.** We will first do some surface theory: sections 1,3,5 of chapter 5, and then use this to prove this theorem.

15. 23/04/2020

Theorem 15.1. *C a smooth projective curve over k. L a line bundle. Then* $H^0(C, L) - H^1(C, L) = 1 - g + \deg(C)$.

Proof. Use exact sequence with skyscrapper sheaf repeatedly. □

We use this to develop intersection theory for curves on smooth projective surfaces.

Let X be a smooth projective surface over $k = \overline{k}$. A curve C on X gives an element in $\mathrm{Div}(X)$.

We say curves meet transversally if they are locally defined by $(f_1), (f_2)$, then f_1, f_2 generate $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Lemma 15.2. *If C is irreducible nonsingular and D meets C transversally, then $|C \cap D| = \deg \mathcal{O}(D)|_C$.*

Theorem 15.3. *There is a unique pairing $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ such that it is linear, commutative, if C, D are transversal then $C \cdot D = |C \cap D|$, and it only depends on the rational equivalence class of D.*

Proposition 15.4. *Let C, D be two curves on X without common components. Then*

$$C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p$$

where the local intersection number is as follows: if they are defined by f_1, f_2 locally, then $(C \cdot D)_p :=$ $\mathrm{lenght}(\mathcal{O}_{X,p}/(f_1, f_2))$.

Proof. We have $0 \to \mathcal{O}(-D)|_C \to \mathcal{O}_C \to \mathcal{O}_{D\cap C} \to 0$. Then $\sum_p (C \cdot D)_p = \chi(\mathcal{O}_{C\cap D}) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}(D)|_C)$. So both sides only depend on the rational equivalence class of D. The same holds for C. And so we can move then to reduce to the case they are transversal. □

Example 15.5. For C nonsingular on X, we have $C^2 = \deg(\mathcal{O}(C)|_C)$. If the ideal sheaf of C is I, then $I/I^2 = \mathcal{O}(-C)|_C$, so $C^2 = \deg((I/I^2)^*) = \deg(N_{C/X})$.

Proposition 15.6. *If C is nonsingular of genus g, then $C \cdot (C + K_X) = 2g - 2 = \deg(K_c)$.*

Theorem 15.7 (Riemann-Roch for surfaces)**.** *D a divisor on X. Then $\chi(X, \mathcal{O}(D)) = \frac{1}{2} D \cdot (D - K_X) + \chi(X, \mathcal{O}_X)$.*

Proof. Write $D \sim C - E$ with C, E nonsingular, then $0 \to \mathcal{O}(D) \to \mathcal{O}(C) \to \mathcal{O}(C)|_E \to 0$. So $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(C)) - \chi(\mathcal{O}(C)|_E) = \chi(\mathcal{O}) + \chi(\mathcal{O}(C)|_C) - \chi(\mathcal{O}(C)|_E)$. Then applying the curve Riemann Roch to $\chi(\mathcal{O}(C)|_C) = C^2 + 1 - g_c$ and $\chi(\mathcal{O}(C)|_E) = C \cdot E + 1 - g_E$. □

Historically, the Riemann–Roch was $\chi(\mathcal{O}) = \frac{1}{12}(K^2 + C_2)$, where C_2 is the second Chern class of X. If $X/\mathbb{C}$, then $c_2 = \chi(X)$.

Theorem 15.8 (Hodge index theorem). *Let H ample on X. D not numerically equivalent to* 0. *Then $D \cdot H = 0 \implies D^2 < 0$.*

Remark 15.9. Let $N(X) = \mathrm{Div}(X)/\equiv$. Then $N(X)$ is finitely generated free $\mathbb{Z}$-module, and the theorem says that the index is $(1, n-1)$.

Lemma 15.10. *If $D^2 > 0$ and $D \cdot H > 0$ for H ample, then for $n \gg 0$ we have $H^0(X, \mathcal{O}(nD)) \neq 0$,*

Proof. By Riemann–Roch, $\chi(C, \mathcal{O}(nD)) > 0$ for $n \gg 0$ if $D^2 > 0$. Now $\dim H^2(\mathcal{O}(nD)) = \dim H^0(\mathcal{O}(K - nD))$ and if $n \gg 0$, $(K - nD) \cdot H < 0$, and so $H^0(X, \mathcal{O}(K - nD)) = 0$. So $\mathcal{O}(KnD)$ has a section.

Now if $D^2 > 0$, choose $H' = D + nH$ ample, and then $H'^2 > 0$, so mH' has a section, which is not true.

Now assume $D^2 = 0$. Choose E with $D \cdot E = 0$. Let $E' = (H^2)E - (E \cdot H)H$, so that $E' \cdot H = 0$. Then let $D' = E' + nD$, by the above case we conclude $(D')^2 = 2n(D \cdot E) + (E')^2$, and we can choose n so that $(D')^2 > 0$, which cannot happen by the above case. □

16. 28/04/2020

Remark 16.1. In the pset we will see that the intersection pairing induces a nondegenerate pairing to $\mathbb{Z}$. The above theorem proves that the index of the pairing is $(1, n-1)$.

16.1. **Monoidal transformation.** For X a smooth projective surface and $p \in X$, the blow up at p $\pi \colon X' \to X$ is called a monoidal transformation.

Then $E = \pi^{-1}(p) \simeq \mathbb{P}^1$, and $E^2 = -1$. This is a (-1) curve.

Proposition 16.2. *We consider the maps $\pi^* \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X')$ and $\mathbb{Z} \to \mathrm{Pic}(X')$ by $n \mapsto E$. Then $\mathrm{Pic}(X') = \mathrm{Pic}(X) \oplus \mathbb{Z}$. Moreover (i) $\pi^*(C) \cdot \pi^*(D) = C \cdot D$, (ii) $\pi^*(C) \cdot E = 0$.*

Corollary 16.3 (Projection formula). $\pi^* C \cdot D = C \cdot \pi_*(D)$.

Proof. We have $\mathbb{Z} \to \mathrm{Pic}(X') \to \mathrm{Pic}(X' \setminus E) \simeq \mathrm{Pic}(X \setminus p)$, and since X is smooth, this is $\mathrm{Pic}(X)$. The pullback splitting this sequence. So it suffices to prove that the image of $\mathbb{Z} \to \mathrm{Pic}(X)$ is $\mathbb{Z}$, but this is injective since $E^2 = -1$.

(i) Choose $C \sim C', D \sim D'$ with C', D' with supports away from p. Then $C' \cdot D' = \pi^*(C') \cdot \pi^*(D')$.

(ii) Similarly, we may assume the support of C doesn't contain p, and then $C \cdot E = \pi^*(C) \cdot E = 0$ since the support of $\pi^*(C)$ does not meet E. □

Theorem 16.4. *$K_{X'} = \pi^* K + E$.*

Proof. We have $K_{X'}|_{X' \setminus E} \simeq K_X|_{X \setminus p}$. So $K_{X'} = \pi^* K_X + nE$. Then $K_{X'} \cdot E = -n$. By the adjunction formula, $(K_{X'} + E) \cdot E = 2g_E - 2 = -2$. This implies $n = 1$. ☐

Corollary 16.5. *$(K_{X'})^2 = (K_X)^2 - 1$.*

Proposition 16.6. *$\pi_* \mathcal{O}_{X'} = \mathcal{O}_X$ and $R^i \pi_* \mathcal{O}_{X'} = 0$ for any $i > 0$. In particular, $H^i(X, \mathcal{O}_X) = H^i(X', \mathcal{O}_{X'})$ for $i \geq 0$.*

Proof. $\mathcal{O}_X \to \pi_* \mathcal{O}_{X'}$ is an isomorphism outside p. Also $R^i \pi_* \mathcal{O}_{X'} = 0$ outside p.

We need to understand them at p. Taking the completion, we can use the formal function theorem for $(R^i \pi_* \mathcal{O}_{X'})_p$. It suffices to prove that $H^i(X'_n, \mathcal{O}_{X'_n}) = 0$ for $i > 0$. Note $\mathcal{O}_{X_0} \simeq \mathcal{O}_{\mathbb{P}^1}$, and that $\pi^{-1}(\mathfrak{m}_p^n) = \mathcal{O}(-nE)$. By the Snake lemma, the kernel $\mathcal{O}_{X'_n} \to \mathcal{O}_{X'_{n-1}}$ is $\mathcal{O}_E \otimes \mathcal{O}(-(n-1)E) = \mathcal{O}_E(n-1)$, which has trivial cohomology for $i > 0$.

Another consequence of the above is that $H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \simeq k[\![x, y]\!] \simeq \hat{\mathcal{O}}_{X,p}$, as we wanted.

From the spectral sequence, we conclude $H^i(X, \mathcal{O}_X) = H^i(X', \mathcal{O}_{X'})$. ☐

Let X be a smooth surface, C an effective Cartier divisor, $p \in C$. Locally around p, C is given by $f_p \in \mathfrak{m}_p$. Let r be such that $f_p \in \mathfrak{m}_p^r \setminus \mathfrak{m}_p^{r+1}$. We denote $r = \mu_p(C)$ the multiplicity of p at C.

Proposition 16.7. *Consider again the blow up $X' \to X$. For a Cartier divisor C, we have $\pi^*(C) = C' + \mu_p(C) E$ where C' is the birational transform of C on X'.*

Proof. Locally in p, C is given by f_p, so $\pi^*(C)$ is given by $\pi^*((f)) = 0$. For coordinates x, y at p, on X' we have $ux = ty$. Writing $f(x, y) = f_r(x, y) + g(x, y)$, for $u \neq 0$ we take $u = 1$ and then $f(ty, y) = y^r(f_r(t, 1) + yh)$. So the zero locus of $f(ty, y)$ is precisely rE locally around E. So we get what we wanted. ☐

Corollary 16.8. *$C' \cdot E = r$ and $p_a(C') = p_a(C) - \frac{1}{2} r(r-1)$ where $r = \mu_p(C)$.*

Proof. The first statement is trivial. For the second, $2p_a(C') - 2 = (K_{X'} + C') \cdot C'$, and we can compute this in terms of C. ☐

Proposition 16.9. *Let C be an irreducible curve on X. Then there exist a finite sequence of blowing ups $X_n \to X_{n-1} \to \cdots \to X_0 = X$ such that C_n is smooth on X_n.*

Proof. If C is not smooth, we blow up its singular point. By the above, this process ends. ☐

Proposition 16.10. *Let $C \subseteq X$. There is a sequence of blow ups $f\colon X_n \to X$ such that: Consider the reduced structure on $f^{-1}(C)$. Then this is normal crossing (there are coordinates $x, y \in \mathfrak{m}_p$ such that it is either $x = 0$ or $xy = 0$).*

Proof. By the previous result, C' is smooth. Let $Y = f^{-1}(C)_{red} = C' \bigcup E_i$.

Now consider blowing up at $p \in Y$. Let $Y_1 = Y' + E$. Then we can compute that $p_a(Y_1) = pa(Y) - \frac{(r-1)(r-2)}{2}$. If $r \geq 3$, then $p_a(Y_1) < p_a(Y_2)$. So we can keep doing blow ups $\mu_p(Y_n) \leq 2$ for all p. Now consider a point with $\mu_p(Y) = 2$. If Y' meets E' in two different points, they are normal crossing. So assume E' and Y' meet at the same point p'. If they are smooth, taking another blowup makes it a triple intersection, so blowing up reduced the arithmetic genus. If they are singular, another blow up drops the arithmetic genus. □

17. 30/04/2020

17.1. **Hartshorne Chapter 5.** Let X, Y projective varieties and $\varphi\colon U \simeq V$ a birational map.

Then there is a maximal open set $U \subseteq X$ such that φ is defined on U. Then consider $\Gamma = \overline{\text{graph}(\varphi|_U)} \subseteq X \times Y$. Then U is the maximal locus such that $p_1\colon \Gamma \to X$ is an isomorphism.

For $p \in X \setminus U$, define $T(p) = p_2 p_1^{-1}(p)$.

Lemma 17.1. *Let φ be a birational map between X and Y. Assume X is normal. Then the locus where φ is not defined is of codimension at least* 2.

Proof. For any codimension 1 point x. the local ring is a DVR, so $\operatorname{Spec} K(X) \subseteq \operatorname{Spec} \mathcal{O}_{X,x}$, and since it is projective over k, there is a lifting $\operatorname{Spec} \mathcal{O}_{X,x} \to Y$. □

Theorem 17.2 (Zariski Main Theorem)**.** *Let f be a birational map between projective X, Y with X normal. Then $T(p)$ is always connected, and of* $\dim \geq 1$ *for $p \notin U$.*

Proof. Consider $p_1\colon \Gamma \to X$. For a fixed p, $p_1^{-1}(p) \subseteq k \times Y$. So $p_2\colon p_1^{-1}(p) \hookrightarrow Y$. So $T(p) \simeq p_1^{-1}(p)$. By Stein factorization, we have $p_1\colon \Gamma \to V \to X$ and since it is birational, $V \to X$ is finite and birational. So $V \simeq X$. Hence $p_1^{-1}(p)$ is always connected.

Now assume $p_1^{-1}(p)$ is 0-dimensional. Then there is $p \in U$ such that $p_1^{-1}(p)$ is 0-dimensional by semicontinuity. Then $p_1^{-1}(U) \to U$ is projective and quasi-finite, which is just being finite. Again, since it is also birational, $p_1^{-1}(U) \simeq U$, so it is defined on f. □

Proposition 17.3. *Let $f\colon X' \to X$ a birational morphism for X', X nonsingular projective surfaces. Consider $p \in X$ which is not defined for f^{-1}. Let $\tilde{X} = \mathrm{Bl}_p(X)$. Then there is a lift $X' \to \mathrm{Bl}_p(X) \xrightarrow{\pi} X$.*

Proof. Consider the birational map g from X' to $\tilde{X}$. Let $q \in X'$ be a point where it is not defined, so $f(q) = p$.

Consider g^{-1}, and consider $T_{g^{-1}}(q)$. By the above, it is connected, and of positive dimension, and it inside $E = \pi^{-1}(p)$. Hence $T_{g^{-1}}(q) = E$. In other words, there exist an open set of E such that g^{-1} is defined on such open set. Let $r \in \tilde{X}$ that maps to q. Then one can do some computations with the blow up to finish. □

Corollary 17.4. *Let $f\colon X' \to X$ a birational morphism between projective smooth surfaces. Let n be the number of irreducible curves in the exceptional locus of f. Then f can be factored through n blow-ups of smooth points.*

Now we investigate whether any birational map between smooth projective varieties X, Y there is a Z and maps $Z \to X$ and $Z \to Y$ that are a sequence of blow ups.

In dimension ≥ 4, we don't know. In characteristic 0, it was proven that there is a sequence of such Z_i.

Theorem 17.5 (Strong factorization theorem for surfaces)**.** *If f from X to X' is a birational map between smooth projective surfaces, then there is Y such that $Y \to X$ and $Y \to X'$ are both sequences of blow-ups.*

Proof. X is projective, so choose H very ample, and $C \in |2H|$ general, so that X doesn't contain the locus where φ is defined in a neighborhood of C. Then it maps to $C' = f(C) \subseteq X'$. Choosing $C_1 \sim C$, then $f(C) \sim f(C_1)$.

Let $m = p_a(C') - p_a(C)$. If $m = 0$, then $C' \simeq C$. If $m > 0$, then C' is singular. Pick $p \in C'$ singular, and blow it up to $\tilde{X}$. Let $\tilde{C}' \subseteq \tilde{X}$ the transform. Then we saw that $p_a(\tilde{C}) < p_a(C')$. Then in this case m decreases. (Need to replace C to a $\tilde{C}$ that is general to $X \to \tilde{X}$)

So we may assume $m \leq 0$. But it is ≥ 0 since $\to C'$ is a normalization. So we may assume $m = 0$, which then means $C \to\simeq f(C)$. Now we claim f^{-1} is a morphism, and then we will be done.

So we have f a birational map from X to X' with $C \simeq f(C)$. If f^{-1} has a fundamental point p, then $T(p)$ is a 1-dimensional curve in X, and as $C \in |2H|$, we have $\#C \cap Z \geq 2$. So $f(p_1) = f(p_2) = p$, whih is a contradiction with $C \to f(C)$ being an isomorphism. □

Theorem 17.6 (Was in midterm)**.** *If E is a curve inside a smooth projective surface X with $E^2 = -1$ and $E \simeq \mathbb{P}^1$, then there is a morphism $X \to Y$ such that $E \mapsto p$ for $p \in Y$ smooth, and this map is the blow-up at p.*

Let X be a smooth projective surface. We look at whether there are $(-1$ curves and blow-down, until there are no more (-1) curves: $X = X_0 \to X_1 \to \cdots$.

This process has to terminate after finitely many steps, since $\mathrm{Pic}(X_i) = \mathrm{Pic}(X_{i+1}) \oplus \mathbb{Z}$ (or that $\mathrm{rank} N^1(X_i) = \mathrm{rank} X^1(X_{i+1}) + 1$). It'll be an exercise that $N^1(X)$ is a free group of finite rank.

Historically, X_n is called a relative minimal surface. Then as a consequence, if $X_n \to Y$ is birational with Y smooth projective, then $X_n \simeq Y$.

There are cases where X_n depend on choices. If it does not, we call it a minimal surface.

The number of (-1) curves on X could also be infinite.

18. 05/05/2020

By a rational curve, we mean a morphism $\mathbb{P}^1 \to X$ such that $f(\mathbb{P}^1)$ is nonconstant.

Theorem 18.1 (Mori)**.** *Let X be a smooth projective variety, with $-K_X$ ample (a Fano variety). Then X has a rational curve. In fact, for any $x \in X$ we can find a rational curve passing through x.*

Lemma 18.2 (Rigidity lemma)**.** *Let $f\colon Y \to Z$ proper surjective. Let $g\colon Y \to X$. Assume f has connected dimension n fibers and that for a $z_0 \in Z$, $g(f^{-1}(z_0))$ is a point in X. Then for any $z \in Z$, $g(f^{-1}(z))$ is also a point.*

Proof. Consider $Y \to X \times Z \to Z$. Let $W \subseteq X \times Z$ the image of Y, with $p\colon W \to Z$ and $h\colon Y \to W$. Then $W_z := p^{-1}(z) = gf^{-1}(z) \subseteq X \times z$. So the assumption is that W_{z_0} is a point. So there is $U \subseteq Z$ such that the fiber of $p^{-1}(U) \to U$ has fibers of dimension 0. Now let $w \in p^{-1}(U)$. Then $h^{-1}(w) \subseteq f^{-1}(p(w))$, which has dimension n, and so $h^{-1}(w)$ also has dimension n. Hence by semicontinuity again, $h^{-1}(w) \geq n$ for all $w \in W$, and this implies that for any $w \in W$, $h^{-1}(w)$ has dimension n.

As $h(f^{-1}(p(w)) = p^{-1}(p(w))$ and f has connected fibers, this implies $h(f^{-1}(p(w)))$ is a single point. □

We defined the notion of two cycles being algebraically equivalent: if we have a $f\colon S \to C$ proper surjective and $g\colon S \to X$, we say $g_*(f^{-1}(p))$ are all algebraically equivalent.

Theorem 18.3 (Bend and Break I). *If X is projective variety, C a smooth curve $g_0\colon X \to X$ nonconstant, consider for a smooth curve D*

$$\begin{array}{ccccc} C & \longleftarrow & C \times D & \xrightarrow{G} & X \\ & & \downarrow & & \\ & & D & & \end{array}$$

And assume (i) $G|_{C\times\{0\}} = g_0$, (ii) $G|_{p\times D} = g_0(p)$ for some $p \in C$, (iii) $G|_{C\times\{t\}}$ is different from g_0 for generic $t \in D$.

Then there is a possible $g_1\colon C \to X$ and a linear combination of rational curves $Z = \sum a_i Z_i$ with $a_i > 0$ such that $(g_0)_*[C]$ is algebraically equivalent to $(g_1)_*[C] + [Z]$ and $g_0(p) \in \bigcup_i Z_i$. In particular, p is contained in a rational curve.

Proof. Consider $\overline{D}$ the projectivization. Let $C \times D \to W \to X$ be the Stein factorization. (iii) means that $\dim W = 2$. So we have a birational map $C \times \overline{D} \to W$. We want to show it is not a morphism.

If it was a morphism, then the rigidity lemma for $C\times\overline{D} \to W$ and $C\times\overline{D} \to C$ would contradict the fact that $C \times \overline{D} \to W$ is birational.

So the birational map $C \times \overline{D} \to W$ has a fundamental point $w \in W$. Let $X \to C \times \overline{D}$ be a common blow ups $h_1\colon X \to C \times \overline{D}$, $h_2\colon X \to W$. Then $h_2(h_1^{-1}(w))$ is 1-dimension and a union of rational curves.

Then letting $1 \in D$ be associated to such w proves the theorem. □

Theorem 18.4 (Bend and Break II). *Let X be projective, $g_0\colon \mathbb{P}^1 \to X$, and $G\colon \mathbb{P}^1 \times D \to X$ for D smooth curve. Assume (i) $G|_{\mathbb{P}^1\times 0} = g_0$, (ii) $G(0 \times D) = g_0(0)$, $G(\infty \times D) = g_0(\infty)$, (iii) $G(\mathbb{P}^1 \times D)$ is a surface.*

Then $(g_0)_*(\mathbb{P}^1)$ is algebraically equivalent to either a reducible curve or a multiple curve, so equivalent to $\sum n_i Z_i$ with $\sum n_i \geq 2$.

Proof. Again we have the birational map $\mathbb{P}^1 \times \overline{D} \to X$, and let S be the common blow up. We will prove the theorem by induction on the number of exceptional curves of $h_1\colon S \to \mathbb{P}^1 \times \overline{D}$.

If it is 0, then we have a contradiction by the rigidity lemma. Let $S' \to \mathbb{P}^1 \times \overline{D}$ be the blow up at one point, say with exceptional curve E. Then $S' \to X$ is well defined along the generic point of E. If $h_2 \colon S' \to X$ is a morphism, then it would suffice to prove $h_1(E)$ and $h_2(E')$ are not constant. If $h_2(E')$ is constant, contract E' to get a surface T and then we have two curves C_1, C_2 that are mapped to a point, and if H is an ample curve, then $(h_2^* H)^2 > 0$, but $h_2^* H \cdot C_i = 0$, and $C_i^2 < 0$, so by the Hodge index theorem, $h_2^* H, C_1, C_2$ are linearly dependent in $NS(T)$. But $\mathrm{rank} NS(T) = 2$ ($T \to \overline{D}$ has fibers $\mathbb{P}^1$) and this gives a contradiction.

If h_2 is not a morphism, then have to consider the cases of where the next blow up is separatedly. If there is no fundamental point on E, then $E \to X$ is nonconstant, and we have another coming from E'. If there is a fundamental point on E: analyze the case it is also on E' or not separatedly.

□

19. 07/05/2020

Theorem 19.1 (Mori)**.** *Let X be smooth projective with $-K_X$ ample. Then for any $x \in X$, we can find a rational curve C through x with $-K_X \cdot C \leq \dim X + 1$.*

Proof. Let C be any smooth projective curve. Then for $f \colon C \to x$, $\dim_{\Gamma(f)} \mathrm{Hilb}(C \times X) \geq \chi(\mathscr{N}_{C/C \times X})$ and $\mathscr{N}_{C/C \times X} \simeq f^* \mathscr{T}_X$, and by Riemann–Roch

$$\chi(C, f^* \mathscr{T}_X) = \deg(f^* \mathscr{T}_X) + (1 - g(C))\mathrm{rank}(f^* \mathscr{T}_X)$$

so $\chi(f^* \mathscr{T}_X) = -K_x \cdot C + (1 - g(C)) \dim X$.

So $\dim_{\Gamma(f)} \mathrm{Hom}(C, X) \geq -K_X \cdot C + (1 - g(C)) \dim X$. But considering the homs that p is mapped to the same point,

$$\dim_{\Gamma(f)}(\mathrm{Hom}(C, X) \geq -K_X \cdot C - g(C) \dim X.$$

So we are done by Break end Bend if $-K_X \cdot C - g(C) \dim X \geq 2$.

If k has characteristic p, then we consider the absolute Frobenius morphism, then we get $\deg(\mathrm{Frob}) \deg(f^*(-K_X)) - g(C) \cdot \dim X$. So if $K = \mathcal{F}_p$, we have the theorem.

Then we do a Lefchetz principle type of thing to prove for the case of characteristic 0: it suffices to prove for a finitely generated $\mathbb{Z}$-algebra R. If $X' / \operatorname{Spec} R$ are the same equations, then X is the base change, and $\operatorname{Spec} k \to \operatorname{Spec} R$ maps to the generic point, so X'_U is smooth and $-K_{X'_U}$ is ample for some U (This is called a spread-out-argument). Now closed points on U all have finite residue fields. So we can find rational curves along the closed points. Now $\mathrm{Hilb}_f(X_U/U) \to U$ is projective if we have finitely many Hilbert polynomials (so then it is finite type), hence if the Hilbert polynomials on the closed points are bounded, then we have $\mathrm{Hilb} \to U$ is dominant, and hence surjective (since is projective). So we have a rational curve passing through the generic point, which is what we needed.

So what remains to do is to prove that if we have a curve passing through x, then there is a curve passing through x with $-K_X \cdot C \leq \dim X + 1$. For this we use Bend and Break II. If C is a rational curve through x and $-K_X \cdot C \geq \dim X + 2$, then the deformation argument before says that there is a family like Bend and Break II, and so we can find another rational curve with smaller $-K_X \cdot C$. □

Theorem 19.2 (Mori)**.** *Let X be smooth projective and let H be ample. Suppose C is irreducible such that $-K_X \cdot C < 0$. Then for any $x \in C$, there is a rational curve E such that* $\dim X + 1 \geq (-E \cdot K_X) > 0$ *and* $\frac{-E \cdot K_X}{E \cdot H} \geq \frac{-C \cdot K_X}{C \cdot H}$.

Proof. The Frobenius trick also works, but using C now. So now we need to prove the second condition.

For finite fields, $\frac{-C \cdot K_X}{C \cdot H} = \frac{-C^{(p)} \cdot K_X}{C^{(p)} \cdot H}$. And repeatedly doing bend and break, $C = C' + Z$ with $-K_X \cdot C' < g(C') \dim X = g(C) \cdot \dim X$. Let $a = -C' \cdot K_X$ and $b = -Z \cdot K_X$ and $c = -C' \cdot H$ and $d = -Z \cdot H$. Then $\frac{a+b}{c+d} = \frac{-C \cdot K_X}{H \cdot K_X} =: M$. For any ϵ, can find a rational curve R irreducible component of Z with $\frac{-K+X \cdot R}{H \cdot R} \geq M_\epsilon$: if $a/c < M$, then $b/d > M$, and then R exists. Look at c when we vary the number of Frobenius. If c is bounded above, then choosing $a + b$ large, $b/d > \frac{a+b}{c+d} - \epsilon$. If c is not bounded, then choose $a/c < M$.

Now keep using break and bend II to satisfy the first condition. Then since the curves in consideration are bounded, the $M - \epsilon$ actually suffices. □

20. 12/05/2020

Definition 20.1. Let $\overline{NE(X)}$ be the closure of the cone which is a $\mathbb{R}$-convex linear combination of effective curves in $N_1(X)$ the dual cone of $NS(X)$ (curves modulo numerical equivalence).

Let X be projective. Then $\overline{NS}(X)$ does not contain a line, as we have am ample divisor H and then $C \cdot H > 0$ by the following.

Lemma 20.2 (Kleiman's criterion)**.** *H is ample on X if and only if H is positive on* $\overline{NE(X)} \setminus \{0\}$.

Proof. If H is ample, it is positive on curves, but we need to show it is positive on the limits. Choose a basis of $N^1(X) \otimes \mathbb{R}$ of the form $H, D_2, \dots, D_n$ with D_i ample and $2H - D_i$ ample. We define a norm by $\sum_{i=1}^n |\cdot D_i|$. If $Z \in \overline{NE(X)}$, then $|Z| = \sum_{i=1}^n Z \cdot D_i$. If $\dim N_1(X) = \rho(X)$, then $2\rho(X) H \cdot Z \geq |Z|$, so $H \cdot Z > 0$ if $Z \neq 0$.

For the converse, we use the notion of Nef: A is nef on X if and only if $A \cdot C \geq 0$ for any effective curve C. We also use the Nakai–Misuhezum criterion: L is ample if and only if for any $Z \subseteq X$ closed, $L^{\dim Z} \cdot Z > 0$.

The if H is positive as above, let D be ample. Then there is m such that $mH - D$ is positive on $\overline{NE(X)} \setminus 0$. In particular, $mH - D$ is nef. Then by the above theorem, a sum of nef and ample is ample. Hence H is ample. □

Theorem 20.3 (Cone Theorem). *Let X be smooth projective. (i) There are countably many rational curves C_i with $0 < C_i \cdot (-K_X) \leq \dim X + 1$ such that $\overline{NE(X)} = \overline{NE(X)}_{K_X \cdot C \geq 0} + \sum \mathbb{R}_{\geq 0} \cdot C_i$, (ii) For any ample divisor H and $\epsilon > 0$, there are finitely many curves C_i such that $C_i \cdot (-K_X) \leq \dim X + 1$ and $\overline{NE(X)} = \sum NE(X)_{(L_X + \epsilon H) \cdot C \geq 0} + \sum_{finite} \mathbb{R}_{\geq 0} C_i$.*

Proof. Let W be the closure of $\overline{NE(X)}_{K_X \cdot C \geq 0} + \sum \mathbb{R}_{\geq 0} \cdot C_i$ for ll C_i with $0 < C_i \cdot (-K_X) \leq \dim X + 1$. We first prove $\overline{NE(X)} = W$. If there is D not in W, then we can choose D to be positive on $W \setminus \{0\}$ but not positive on the whole $\overline{NE(X)}$. Let H be ample and μ be maximal such that $H + \mu D$ is nef. Then there is Z such that $Z \cdot (H + \mu D) = 0$, so $Z \cdot D < 0, Z \cdot K_X < 0$. As Z is a limit, write it as $Z_k \to Z$ and $Z_i = \sum a_{ij} Z_{ij}$ with Z_{ij} effective. For $\mu' < \mu$, $H + \mu' D$ is ample. Then there is Z_{kj} such that

$$\frac{-Z_k \cdot K_X}{Z_k \cdot (H + \mu' D)} \leq \frac{-Z_{kj} \cdot K_X}{Z_{kj} \cdot (H + \mu' D)}$$

and so we can apply Bend and Break so that there is a rational curve E_k such that $0 < E_k \cdot (-K_X) \leq \dim X + 1$ and $\frac{E_K \cdot (-K_X)}{E_K \cdot H + \mu' D} \geq \frac{-Z_K \cdot K_X}{Z_K \cdot (H + \mu' D)}$. But now $\frac{E_K \cdot (-K_X)}{E_K \cdot (H + \mu' D)} \leq \frac{E_K \cdot (-K_X)}{E_K \cdot H} \leq M$ for some constant M, as $E_K \in W$. Combining everything,

$$\frac{-Z_K \cdot (K_X)}{Z_k \cdot (H + \mu' D)} \leq M,$$

and letting $k \to \infty$ and $\mu' \to \mu$ gives a contradiction.

One can continue to prove (ii) and prove we don't need to take the closure. □