18.906: ALGEBRAIC TOPOLOGY II, SPRING 2020

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Website

Ofice Hours: Tuesdays 3–5.

1. 03/02/2020

Agenda:

- Basic homotopy theory - category theory and obstruction theory
- Vector bundles, principal bundles, ...

- Spectral sequences (Serre)
- Characteristic classes, applications

1.1. **Some category theory.** Talked about colimits.

Example 1.1. For a group G, consider it as a category. A group action is a functor $G \to \text{Set}$. Then the colimit of this functor is the orbit space $G\backslash X$.

Adjoints and stuff

2. 05/02/2020

Missed lecture (STAGE)

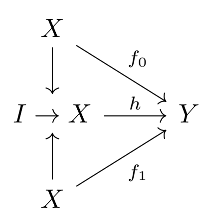
3. 07/02/2020

3.1. **Homotopy category.**

Proposition 3.1. *k-spaces are closed under quotients, closed subspaces, product with compact Hausdorff spaces and colimits. Moreover, $k(X \times_{\text{Top}} Y) = X \times_{\text{kTop}} Y$.*

Definition 3.2. A space is *weakly Hausdorff* if the image of $X \xrightarrow{\Delta} X \times_{\text{kTop}} X$ is closed. Weakly Hausforff k-spaces are also Cartesian closed.

Example 3.3. For $f_0, f_1 \colon X \to Y$, we say they are homotopic if there is an extension



and a map $h \colon I \times X \to Y$ is the same as a map $I \to Y^X$, and so $[X, Y] = \pi_0(Y^X)$.

Definition 3.4. A *pointed category* is a category with an initial and terminal object that are isomorphic.

Note that if a category was Cartesian closed and pointed, then $\mathscr{C}(X, Y) = \mathscr{C}(*, Y^X) = \mathscr{C}(\emptyset, Y^X) = *$. So Top_* will not be Cartesian closed.

But we still can make sense of $Y^X* = \{f \colon X \to Y, f(*) = *\} \subseteq Y^X$. And now we have:

Proposition 3.5. $\mathrm{Top}_*(W, Y_*^X) \simeq \mathrm{Top}_*(W \wedge X, Y))$ *where* $W \wedge Y = (W \times Y)/(W \vee Y)$ *is the smash product.*

It behaves like a tensor product. For instance, we have $W \wedge S^0 = W$, but associativity is false in Top_*, but true in kTop_*.

For a pair $(X, A) \in \mathrm{Top}$, we can make sense of $X/A \in \mathrm{Top}_*$, and taking $A = \emptyset$, we must have $X/\emptyset = X_+ := X \sqcup *$, and this functor $X \mapsto X_+$ is left adjoint to the forgetful functor.

We think of spheres as $S^m = I^m/\partial I^m$, and we can compute $S^m \wedge S^n = S^{n+m}$.

Definition 3.6. The *reduced suspension* of X is $\Sigma X := S^1 \wedge X$. The *loop space* of X is $\Omega X := X_*^{S^1}$. We note that $\Sigma^n X = S^n \wedge X$ and $\Omega^n X = X_*^{S^n}$.

Theorem 3.7 (Milner)**.** *If X is a pointed countable CW complex, the ΩX is homotopic equivalent to a pointed countable CW complex.*

Definition 3.8. The category HoTop_* is where the hom sets $[\cdot, \cdot]_*$ are modded by pointed homotopies. We define $\pi_n(Y) := [S^n, Y]_*$. This is the same as $\pi_1(\Omega^{n-1} Y)$ if $n > 0$, and so it has a group structure.

Theorem 3.9 (Today)**.** *If X is simply connected finite complex and all $\pi_*(X)$ are known, then $X \simeq *$.*

4. 10/02/2020

4.1. **Fiber bundles and Fibrations.**

Definition 4.1. A *fiber bundle* is a continuous map $p \colon E \to B$ such that for any $b \in B$, there is an open neighborhood $b \in U$ such that $p^{-1}(U) \simeq p^{-1}(b) \times U$. We call E the *total space*, B the *base space*, p the *projection*.

Example 4.2. A covering space is a fibre bundle with discrete fibers.

Example 4.3 (Hopf bundle)**.** $S^3 \subseteq \mathbb{C}^2$, and we have a map to $\mathbb{CP}^1 \simeq S^2$ taking v to $\mathbb{C}v$.

Example 4.4 (Stiefel Manifold and Grassmannian)**.** $V_k(\mathbb{R}^n)$ is the space of ordered orthonormal k-frames in $\mathbb{R}^n$. It is a compact manifold (sits inside $(S^{n-1})^k$). It is also $V_k(\mathbb{R}^n) \simeq \mathrm{Hom}_{\mathrm{isometric}}(\mathbb{R}^k, \mathbb{R}^n)$. $\mathrm{Gr}_k(\mathbb{R}^n)$ is the space of k dimensional linear subspaces. The map $\mathrm{span} \colon V_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ is a fibre bundle.

Note the Hopf bundle is the special case $(k, n) = (1, 2)$.

We will also talk about Lie groups: $O(n), SO(n), U(n), SU(n), Sp(n)$.

Theorem 4.5 (Hilbert's 5th problem)**.** *If G is a topological group and homeomorphic to a finite CW complex, then G is a compact Lie group.*

If G is a topological group and homotopic equivalent to a finite complex, we define $G \sim H$ generated by homomorphisms that are homotopic equivalent.

Theorem 4.6. *There are uncountably many non-equivalent topological groups homotopic equivalent to S^3.*

Theorem 4.7. *A smooth map $p\colon E \to B$ is a fibre bundle provided that: (i) $p^{-1}(b)$ is compact for all b (we call this proper), (ii) The map $\mathrm{d}p$ is surjective at every point (we call this a submersion).*

Corollary 4.8. *For G a compact Lie group, and $K \subseteq H \subseteq G$ closed subgroups. Then $G/K \to G/H$ is a fiber bundle.*

Note that the example $V_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ is in fact $O(n)/(O(n-k) \times I_k) \to O(n)/(O(n-k) \times O(k))$ is an example of the above.

5. 14/02/2020

Talked about fibrations last time. We denote $X \times_Y E = f^*E$ the pulback of $E \to Y$ along $f\colon X \to Y$.

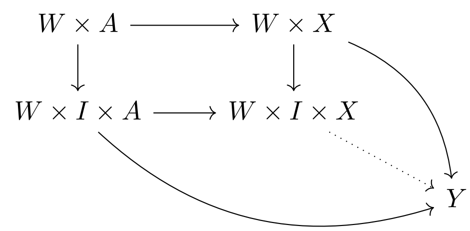
Proposition 5.1. *Suppose $f_0 \sim f_1\colon X \to Y$, and a fibration $p\colon E \to Y$. Then $f_0^*E \simeq f_1^*E$.*

Proof.

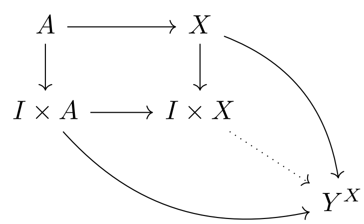
$$\begin{array}{ccccc}
f^*E & \longrightarrow & \mathrm{ev}^*E & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\
X & \longrightarrow & Y^X \times X & \xrightarrow{\mathrm{ev}} & Y \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \\
* & \xrightarrow{f} & Y^X & &
\end{array}$$

and then use that a path in Y^X give the homotopy we want. ∎

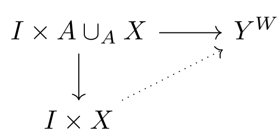
5.1. **Cofibrations.** For a map $i\colon A \to X$, when is $Y^X \to Y^A$ a fibration? The adjoint of the path-lifting diagram becomes



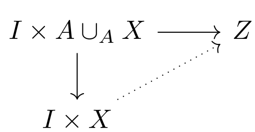
Doing an adjunction again, we get



and now this is the same as



Definition 5.2. $i\colon A \to X$ is a *cofibration* if it satisfies the homotopy extension property:



We proved the following lemma.

Lemma 5.3. *If $i\colon A \to X$ is a cofibration, then for any Y we have that $Y^X \to Y^A$ is a fibration.*

There is a universal example: denoting $M(i) = I \times A \cup_A X$, the universal example is $Z = M(i)$. So i is a cofibration if and only if $M(i) \to I \times X$ admits a retract.

Example 5.4. $S^{n-1} \to D^n$.

Example 5.5 (Preservation of cofibrations)**.** Pushouts: for $A \to B$, $B \to B \cup_A X$ is a cofibration. Coproducts (disjoint union). Composition. Product with any space (dual to fibrations being preserved under exponentials).

Proposition 5.6. *If $A \subseteq X$ is a sub CW complex, then it is a cofibration.*

Remark 5.7. Cofibrations are usually closed inclusions.

Proposition 5.8. *If $i\colon A \to X$ is a cofibration and $A \sim *$, then $X \sim X/A$.*

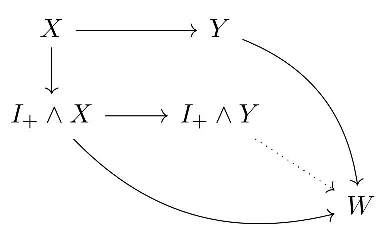
We now have the following easy proposition.

Proposition 5.9. *Any map $X \to Y$ admits a factorization $X \to M(f) \to Y$ where $X \to M(f)$ is a cofibration and $M(f) \to Y$ is a homotopy equivalence, and this can be done naturally in f.*

6. 18/02/2020

6.1. **Barratt–Poppe periodicity.** Here we work with everything pointed. This changed slightly the notions of fibration and cofibration.

A cofibration becomes



and we can see that a cofibration is a (regular) cofibration that preserves basepoint.

The mapping cylinder $M(f)$ must be changed by colapsing the base points, and this becomes the mapping cone $C(f)$. Also note that $C(f)$ is the pushout of $X \to Y$ and $X \to C(X \to *)$. It is an example of a homotopy colimit. This means $Y \to C(f)$ is a cofibration. If f is already a cofibration, then so is $C(X \to *) \to C(f)$. Then $C(f) \simeq C(f)/C(X \to *) = Y/X$.

Example 6.1. $\Sigma X = C(X \to *)$.

The cone also has a universal property: $X \xrightarrow{f} Y \to C(f)$ is null-homotopic with a canonical homotopy $I_+ \wedge X \to C(f)$. It is universal among such null-homotopies. So if $[C(f), Z]_* \to [Y, Z]_* \to [X, Z]_*$ is "exact", in the sense that things that map to a null-homotopy are exactly the image. We call $X \to Y \to C(f)$ co-exact.

We can take the "cone resolution" of any map $f\colon X \to Y$. Since $Y \to C(f)$ is a cofibration, $C^2(f)$ is homotopic to $C(f)/Y = \Sigma X$. In the same way, $C^3(f)$ is homotopic to ΣY. We note that the map $\Sigma X \to \Sigma Y$ is $-\Sigma(f)$. We get

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y \to \Sigma C(f) \to \Sigma^2 X \to \cdots .$$

This is the homotopic exact sequence of a pair: if $A \subseteq X$, then we have $H_*(X, A) = \overline{H_*}(C(i))$ since $\overline{H_*}(X \cup CA) = H_*(X \cup CA, *) = H_*(X \cup CA, CA) = H_*(X \cup C_{\leq 1/2}A, C_{\leq 1/2}A) = H_*(M(i), A \times I) = H_*(X, A)$.

7. 21/02/2020

We had $\pi_n(X) = [(I^n, \partial I^n), (X, *)]$. We want to define $\pi_n(X, A, *)$.

Definition 7.1. Let $J_n = \partial I^{n-1} \times I \cup I^{n-1} \times \{0\}$. We define $\pi_n(X, A, *) := [(I^n, \partial I^n, J_n), (X, A, *)]$.

Note that we have a map $\partial \colon \pi_n(X, A) \to \pi_{n-1}(A)$.

This gives us a "sequence" (compositions are trivial):

$$\cdots \to \pi_2(X, A) \to \pi_1(A) \to \pi_1(X) \to \pi_1(X, A) \to \pi_0(A) \to \pi_0(X).$$

of pointed sets.

7.1. **Homotopy fibers.** A pointed fibration is

$$\begin{array}{ccc} W & \longrightarrow & E \\ {\scriptstyle \mathrm{in}_0}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ I_+ \wedge W & \longrightarrow & B \end{array}$$

And we also have a factorization

$$\begin{array}{ccc} X & \xrightarrow{\text{hom.equiv}} & T(f) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where $T(f) = \{(x, w) \in X \times Y^I \colon w(1) = f(x) \in Y\}$. We call $F(f) = p^{-1}(*)$ the homotopy fiber: $F(f) = \{(x, w) \in X \times Y^I \colon w(1) = f(x) \in Y, w(0) = *\}$. This is the same as a pullback

$$\begin{array}{ccc} F(f) & \longrightarrow & T(f) \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & Y \end{array}$$

or a pullback

$$\begin{array}{ccc} F(f) & \longrightarrow & X \\ \downarrow & & \downarrow \\ P(Y) & \xrightarrow[\text{fib}]{} & Y \end{array}$$

where $P(Y) = T(* \to Y) = \{w \colon I \to Y \colon w(1) = *\}$, which is contractible.

Note that the fibre of $P(Y) \to Y$ is ΩY, which means that the fiber of the fibration $F(f) \to X$ is also ΩY

Lemma 7.2 (Was in homework). *For $g\colon W \to E$ and $f\colon W \to B$ such that $p \circ g \sim f$, then there is $g' \sim g$ with $p \circ g' = f$.*

So $p^{-1}(*) \to E \to B$ is exact, that is, $[W, F]_* \to [W, E]_* \to [W, B]_*$ is exact for all W. As before, we get a sequence

$$Y \xleftarrow{f} X \leftarrow F(f) \leftarrow \Omega Y \leftarrow \Omega X \leftarrow \Omega F(f) \leftarrow \Omega^2 Y \leftarrow \cdots$$

and hence an exact sequence

$$\pi_0(Y) \leftarrow \pi_0(X) \leftarrow \pi_0(F(f)) \leftarrow \pi_1(Y) \leftarrow \pi_1(X) \leftarrow \pi_1(F(f)) \leftarrow \pi_2(Y) \leftarrow \cdots$$

Lemma 7.3. *For a pair $i\colon A \hookrightarrow X$, there is an isomorphim of $\pi_{n-1}(F(i))$ with $\pi_n(X, A)$ making the following diagram commute*

$$\begin{array}{ccc} \pi_n(X) & \longrightarrow & \pi_n(X,A) \\ \sim\downarrow & & \downarrow\sim \\ \pi_{n-1}(\Omega X) & \longrightarrow & \pi_{n-1}(F(i)) \end{array}$$

and the right vertical arrow is compatible with the maps to $\pi_{n-1}(A)$.

Corollary 7.4. *$\pi_n(X, A)$ is a group for $n \geq 2$, and the exact sequence before is exact.*

Also, $\pi_1(A)$ acts on $\pi_n(X, A)$ for all $n \geq 1$ compatibly with the long exact sequence. The action is given by connecting J_n with a larger J_n with the element of $\pi_1(A)$. So the long exact sequence is $\pi_1(A)$ equivariant.

8. 24/02/2020

8.1. **Techniques in CW complexes.**

Definition 8.1. A *relative CW complex* is a space X with a filtration $A = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X$ such that: (1) for all $n \geq 0$ there is a pushout square

$$\begin{array}{ccc} \coprod S^{n-1} & \xrightarrow{\text{attaching}} & Xn-1 \\ \downarrow & & \downarrow \\ \coprod D^n & \xrightarrow[\text{characteristic}]{} & X_n \end{array}$$

Definition 8.2. A map $p\colon E \to B$ has the relative lifting property with respect to $A \to X$ if

$$\begin{array}{ccc} I \times A \cup 0 \times X & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ I \times X & \longrightarrow & B \end{array}$$

Definition 8.3. A *Serre fibration* is $p\colon E \to B$ satisfying the homotopy lifting property for all CW complexes.

Remark 8.4. The previous fibration is called a *Hurewicz fibration.*

Lemma 8.5. *Let $p\colon E \to B$ be an acyclic (weak equivalence) Serre fibration. Then every fiber is weakly contractible (there is a weak equivalence to a point).*

Proof. We may assume E, B are path-connected. Then from the long exact sequence we get what we want. □

Proposition 8.6. *$p\colon E \to B$. Then the following are equivalent: (1) p is a Serre fibration, (2) p has the homotopy lifting property for all disks, (3) p has the relative homotopy lifting with respect to $S^{n-1} \hookrightarrow D^n$, (4) p has the relative homotpy lifting with respect to relative CW inclusions $A \hookrightarrow X$.*

Proposition 8.7 (Relative straightening)**.** *Consider*

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ \downarrow & & \downarrow \text{Serre} \\ X & \xrightarrow[g]{} & B \end{array}$$

Assume there is $g' \sim g$ making the diagram commute. If g' admits a filler, then so does g.

Theorem 8.8 (Whitehead's little theorem)**.** *$f\colon X \to Y$ is a weak equivalence if and only if $[W, X] \to [W, Y]$ is a bijection for all W CW complexes.*

For this we will use

Theorem 8.9 (Basic lifting theorem)**.** *We have*

$$\begin{array}{ccc} A & \longrightarrow & E \\ \text{rel. CW} \downarrow & \nearrow & \downarrow \text{acyclic Serre} \\ X & \longrightarrow & B \end{array}$$

Proof. By induction, we need to prove

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & E \\ \big\downarrow & \nearrow & \big\downarrow \text{acyclic Serre} \\ D^n & \xrightarrow[g]{} & B \end{array}$$

In the case $B = *$, we need to prove that if E is weakly contractible, then $S^{n-1} \to E$ extends to D^n. We use the fact that if X is path connected, then $\pi_1(X,*)\backslash\pi_n(X,*) \simeq [S^n, X]$.

For the general case, we change g to g' such that $g' = g \circ \varphi$ where $\varphi\colon D^n \to D^n$ is $\varphi(v) = \max(0, 2|v| - 1)v$. By the relative straightening, it suffices to prove for g', and we replace g by g'.

Now g is constant on $D^n_{1/2} \to B$. Let $W = I \times S^{n-1}$ be the annulus. Then we have

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & E \\ \big\downarrow & \overset{h}{\nearrow} & \big\downarrow \\ I \times S^{n-1} & \longrightarrow & B \end{array}$$

And now $S^{n-1}_{1/2} \xrightarrow{h} E \xrightarrow{p} E$ is constant, so $h(S^{n-1}_{1/2})$ lands on the fiber $F = p^{-1}(g(0))$. But F is weakly contractible, so h extends to a map $D^n_{1/2}$. This gives the whole map from D^n. □

Proof of Whitehead's little theorem. Consider $X \to Tf \to Y$ the factorization into a homotopy equivalence and a Hurwevicz fibration. This implies $Tf \to Y$ if a weak equivalence. Since $[W, X] \simeq [W, Tf]$, we may assume that our original map is a acyclic Serre fibration.

Then the surjection follows from the previous lemma on the relative CW inclusion $\emptyset \to W$, and the injectivity follows from the previous lemma on the relative CW inclusion $\partial I \times W \to I \times W$. □

9. 28/02/2020

9.1. **Conectivity, Skeletal approximation, CW approximation.**

Definition 9.1. A space X is n-connected if $\pi_q(X) = 0$ for $q \leq n$.

Definition 9.2. A pair (X, A) is n-connected if: $n = 0$: $\pi_0(A) \twoheadrightarrow \pi_0(X)$; $n > 0$: $\pi_q(X, A, *) = 0$ for all $q \leq n$.

Definition 9.3. $f\colon X \to Y$ is an n-equivalence if: $n = 0$: $\pi_0(X) \twoheadrightarrow \pi_0(Y)$; $n > 0$: $\pi_q(X) \to \pi_q(Y)$ is iso if $q < n$ and surjective if $q = n$.

Theorem 9.4 (Whitehead)**.** *Let $X \to Y$ be an n-equivalence. Then $[W, X] \to [W, Y]$ is iso if* $\dim W < n$ *and surjective if* $\dim W = n$.

Theorem 9.5 (Skeletal approximation)**.** *Consider $f\colon (X,A) \to (Y,B)$ a relative CW complex map. Then f is homotopic relative to A to a skeletal map. Moreover, if f_0, f_1 are skeletal and homotopic, then the homotopic can be deformed so as to send X_n into Y_{n+1}.*

Example 9.6. Consider $Y = S^n$ and $X = S^q$, with $S^n = * \cup D^n$. Then if $q < n$, it is homotopic to the identity. Hence $\pi_q(S^n) = 0$ for $q < n$.

Moreover, we also have the following as a consequence of the skeletal approximation.

Theorem 9.7. *(X, X_n) is n-connected.*

Theorem 9.8 (CW approximation)**.** *For any Y there exist a CW complex X and a weak equivalence $X \to Y$.*

Proof. We prove by induction. We will construct the n-equivalence $X_n \to Y$ inductively. For $n = 0$ take a point for each path component to be X_0. Now we may assume Y is path-connected. Pick generators of $\pi_1(Y)$ as a group, and for each one put the loop into X_1. Now we want to make the isomorphism on π_1, so we take generators for the kernel $\pi_1(X_1) \to \pi_1(Y)$. Then we have $\coprod S^1 \to X_1$, so we can pushout the $\coprod S^1 \to \coprod D^2$ to get $X_2' \to Y$. Then the π_1 agree. Now do the same to construct X_2 (making π_2 surjective): choose generators of $\pi_2(Y)$ as a $\mathbb{Z}\pi_1(Y)$-module and attach them. Continue this way, use skeletal approximation for the argument of putting in the relations. $\square$

Theorem 9.9 (CYC Wall)**.** *Assume Y is simply connected, and $H_n(Y)$ is finitely generated for all n. Then there exist a weak equivalence $X \to Y$ with X a CW complex where there are $\beta_n + \tau_{n-1}$ n-cells in X. (β_n the betti numbers, and τ_n the number of generators in the torsion of homology).*

10. 02/03/2020

10.1. **Postnikov systems.**

Proposition 10.1. *Suppose (X, A) is a relative CW complex, with all cells with dimension $> n$. Then (X, A) is n-connected.*

Theorem 10.2. *Let X be a space and $n \geq 0$. There exist a map $f\colon X \to X[n]$ such that (i) $\pi_q(X, *) \to \pi_q(X[n], f(*))$ is an isomorphism for all $* \in X$ and all $q \leq n$, (ii) $\pi_q(X[n], f(*)) = 0$ for all $q > n$, (iii) $X[n]$ is built from X by attaching cells in dimension $\geq n + 2$.*

$X[n]$ is a *Postnikov section* of X.

Example 10.3. If $n = 0$, then $X[0] = \pi_0(X)$ with discrete topology. If $n = 1$ and X is path-connected, $X[1]$ is a $K(\pi_1(X), 1)$. For $n > 1$, we have new spaces.

Proof of the theorem. We build it via skeletons $X = X(n) \to X(n+1) \to \cdots$ such that $\pi_i(X(i)) = 0$. We construct $X(i) \to X(i+1)$ by attaching $i+2$ cells. Then $\pi_q(X(i)) \xrightarrow{\sim} \pi_q(X(i+1))$ for $q \le i$ and $\pi_{i+2}(X(i+1), X(i)) \to \pi_{i+1}(X(i)) \to \pi_{i+1}(X(i+1)) \to 0$. So we can make $\pi_{i+1}(X(i+1)) = 0$ by picking generators of $\pi_{i+1}(X(i))$ attaching a $i+2$ cell. Then $X[n] = \varinjlim_{i \ge n} X(i)$, and we need to justify why $\varinjlim \pi_q(X(i)) = \pi_q(X[n])$, which is a consequence of the next proposition. □

Proposition 10.4. *Let X be a CW complex an pick a cell structure. Then any compact subset of X lie in a finite subcomplex.*

Proof. X is a disjoint union of cell interiors as a set. Note that any subset that meet each cell interior in a finite set is discrete. So any compact set meet finitely many interiors (otherwise choose a point in each of infinitely, and will be a subset of the compact sequence which is discrete and infinite). This also proves that a CW complex is finite if and only if is compact.

Now we prove each cell lies in a finite subcomplex, which will finish the problem. A boundary of a cell is a compact subset of dimension lesser, so it meets finitely many cells, which are all contained in a finite subcomplex by induction. □

We may wonder how natural/unique $X[n]$ is.

Lemma 10.5. *Let $n \ge 0$, suppose Y is such that $\pi_q(Y, *) = 0$ for all $* \in Y$ and $q > n$. Let (X, A) be a relative CW complex with cells in dimension $\ge n+2$. Then $[X, Y] \to [A, Y]$ is bijective.*

As a consequence, for $X \to Y$ and any choice of $X[m]$, $Y[n]$, with $m \ge n$ there is a unique diagram as follows up to homotopy.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X[m] & \longrightarrow & Y[n] \end{array}$$

This also implies that the $X[n]$ themselves are unique up a unique homotopy class of weak-equivalences.

This also implies that $X[n+1] \to X[n]$ is unique up to homotopy. So we have a tower $X \to \cdots \to X[2] \to X[1] \to X[0]$. This is called the *Postnikov tower*. It is sort of a dual of a skeletal filtration.

11. 06/03/2020

From last time:

Corollary 11.1. *X is simply connected with $\overline{H}_q(X) = 0$ for $q < n$. Then X is $(n-1)$-connected.*

Example 11.2 (Non-example)**.** Poincaré 3-sphere. Consider the isosahedral group $I \subseteq SO(3) \subseteq SU(2)$, and then lifting I to $\tilde{I}$, we consider $P = S^3/\tilde{I}$. Then $\pi_1(P) = \tilde{I}$, and since $\tilde{I}^{ab} = 1$, we have $H_1 = 0$ and by Poincaré duality $H_2 = 0$.

11.1. **EM spaces.**

Our model was: for an abelian group π, take a free resolution $0 \to F_1 \to F_0 \to \pi \to 0$ and then construct the cell complex M (Moore space) like that, and then $K(\pi, n) = \tau_{\leq n} M$.

Lemma 11.3. *Let Y be any space with $\pi_q(Y) = 0$ for $q \neq n$ and $\pi_n(Y) = G$. Then $[\tau_{\leq n} M, Y]_* \xrightarrow{\pi_n} \mathrm{Hom}(\pi, G)$ is an isomorphism.*

Corollary 11.4. *If $\pi = G$, then there is a weak equivalence $\tau_{\leq n} M \to Y$. If both are CW complexes, then there are homotopic. So there is a functor $\mathrm{Ab} \to \mathrm{Ho}(CW_*)$ with a section by π_n.*

Proof. Consider the co-exact sequence $\bigwedge S^n \to \bigwedge S^n \to M \to \bigwedge S^{n+1} \to \bigwedge S^{n+1}$ and take $[\cdot, Y]_*$ for a Y as above. Then we get

$$\mathrm{Hom}(F_1, G) \leftarrow \mathrm{Hom}(F_0, G) \leftarrow [M, Y]_* \to 0,$$

and so $[M, Y]_* = \mathrm{Hom}(\pi, G)$. Then $[M, Y]_* = [\tau_{\leq n} M, Y]_*$. □

Remark 11.5. The Moore space cannot be made functorial.

If $n = 1$, π, G can be non-abelian.

11.2. **Relation with cohomology.**

Write $K = K(\pi, n)$. Then $\overline{H}_q(K, G) = 0$ for $q < n$. Then

$$0 \to \mathrm{Ext}(H_{q-1}(K), G) \to H^q(K, G) \to \mathrm{Hom}(H_q(X), G) \to 0.$$

So $\overline{H}^q(K,G)=0$ for $q<n$, and for $q=n$ we get $H^n(K,G)=\mathrm{Hom}(H_n(K),G)$ and by Hurwewicz we have $H_n(K)=\pi$. Hence

$$H^n(K,G)\xrightarrow{\sim}\mathrm{Hom}(\pi,G).$$

In the case $G=\pi$, we have a distinguished $\iota_n\in H^n(K,\pi)$ given by the identity map under the above identification. This is called the *fundamental class.*

If X is any space, and we have $f\colon X\to K$, we can consider $f^*(\iota_n)\in H^n(X,\pi)$. This gives

$$[X,K(\pi,n)]\to H^n(X,\pi).$$

Theorem 11.6. *If X is a CW complex, then $[X,K(\pi,n)]\xrightarrow{\sim}H^n(X,\pi)$ as abelian groups. i.e. ι_n is the universal n-dimensional cohomology class with coefficients in π.*

Example 11.7. $H^2(X,\mathbb{Z})=[X,\mathbb{C}P^\infty]$, $H^1(X,\mathbb{Z})=[X,\mathbb{R}P^\infty]$, $H^1(X,\mathbb{Z})=[X,S^1]$.

Proof. We work with pointed maps $[X,K]_*$, and we need to prove $[X,K]_*\xrightarrow{\sim}\overline{H}^n(X,\pi)$. The group structure on the left is given by the following: $K(\pi,n)$ is weak equivalent to $\Omega^2K(\pi,n+2)$, and this gives the group structure. To see this is a group homomorphism, note that

$$\begin{array}{ccccc}\overline{H}^n(X)\times\overline{H}^n(X) & \longrightarrow & & \longrightarrow & \overline{H}^n(X)\\ \downarrow\sim & & & & \sim\uparrow\\ \overline{H}^{n+1}(\Sigma X)\times\overline{H}^{n+1}(\Sigma X) & \longrightarrow & \overline{H}^{n+1}(\Sigma X\wedge\Sigma X) & \longrightarrow & \overline{H}^{n+1}(\Sigma X)\end{array}$$

ia the map $\Sigma X\to\Sigma X\wedge\Sigma X$ by pinching. This is the same way as the multiplication on the left is given.

Now to prove it is an isomorphism, work by induction: $\bigwedge S^{n-1}\to X_n\to X_{n+1}$ is co-exact, so gives a exact sequence under both H^* and $[-,K]_*$. This reduces to proving it for wedges of spheres. This is true by the definition of $K(\pi,n)$.

Then we do a limiting argument if the CW complex is infinite. □

Remark 11.8. (i) Could prove that H^n is representable directly: by axiomatizing certain properties of H^n: (Brann) representability.

(ii) We used that $K(\pi,n)$ was weak equivalent to $\Omega K(\pi,n+1)$ and that is represents the suspension isomorphism in H^*. A sequence of pointed spaces E_n with maps $\Sigma E_n\to E_{n+1}$ is a "spectrum". This is a Ω-spectrum if the adjoints $E_n\to\Omega E_{n+1}$ are weak-equivalences. With such a spectrum, we define

$$\overline{E}^n\colon \mathrm{Ho}(\mathrm{Top}_*)\to\mathrm{Ab},\quad X\mapsto[X,E_n]_*$$

a *generalized Cohomology theory* (satisfy all axioms except dimension). And any generalized cohomology theory arrises in such a way.

(iii) From the Yoneda lemma, we get that natural transformations from $H^m(-,\pi)$ to $H^n(-,G)$ is the same as $H^n(K(\pi,m),G)$. As an example, $H^*(K(\mathbb{F}_2,n),\mathbb{F}_2)$ give the optimal value category for $H^*(-,\mathbb{F}_2)$ (Steenrod operations).

12. 09/03/2020

12.1. **Obstruction Theory.** Recall that for a relative CW complex (X,A), we define $C_n(X,A) = H_n(X_n, X_{n-1})$, and the boundary map is

$$H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2}).$$

And fixing a set of n-cells K_n, we have $C_n(X,A) \simeq \mathbb{Z}K_n$.

Theorem 12.1. *$H_*(X,A) = H_*(C_*(X,A))$ and $H^*(X,A,\pi) = H^n(\mathrm{Hom}(C_*(X,A),\pi))$.*

Recall that we saw $[X,Y] = *$ if $\dim X \leq n$ and Y is n-connected. A generalization of this question is when we can extend $f\colon A \to Y$ to $X \to Y$.

Note that we can always extend to X_0. To extend to X_1, we need endpoints of 1-cells to map to the same connected component, and sometimes we can change the choice on X_0 to allow it.

Say we want to extend $f\colon X_n \to Y$ to X_{n+1}. We assume that Y is simple and path-connected, as then $[S^n, Y] = \pi_n(Y)$. Then we get $K_{n+1} \to \pi_n(Y)$ and so $C_{n+1}(X,A) \to \pi_n(Y)$, hence an element of $\theta_f \in C^{n+1}(X,A,\pi_n(Y))$. This is called the *obstruction cocycle.* Now f extends to $X_{n+1} \to Y$ if and only if $\theta_f = 0$.

Proposition 12.2. *θ_f is a cocycle.*

Proof. We need to prove $H_{n+2}(X_{n+2}, X_{n+1}) \xrightarrow{\partial} H_{n+1}(X_{n+1}) \to H_{n+1}(X_{n+1}, X_n) \xrightarrow{\theta_f} \pi_n(Y)$ is zero. Now $H_{n+2}(X_{n+2}, X_{n+1})$ actually all come from $\pi_{n+2}(X_{n+2}, X_{n+1}, *)$, and we have a corresponding sequence of maps in homotopy mapping to this. Now θ_f is defined by $\pi_{n+1}(X_{n+1}, X_n, *) \to \pi_n(X_n) \xrightarrow{f_*} \pi_n(Y)$. But now along the way we have the map $\pi_{n+1}(X_{n+1}) \to \pi_{n+1}(X_{n+1}, X_n) \to \pi_n(X_n)$, which is 0. $\square$

Theorem 12.3 (Main theorem of obstruction theory)**.** *Let (X,A) be a relative CW complex, Y simple and path-connected. Suppose we have $f\colon X_n \to Y$. Then $f|_{X_{n-1}}$ extends over X_{n+1} if and only if $\theta_f \in H^{n+1}(X,A,\pi_n(Y))$ is trivial.*

Proof. Consider two maps $f_0, f_1 \colon X_n \to Y$ that are homotopic when restricted to X_{n-1}. We want to relate θ_{f_0} and θ_{f_1}. Our data is the same as a map $g \colon X_n \times \partial I \cup X_{n-1} \times I \to Y$.

For a CW complex X', we give $X' \times I$ a cell structure: each i cell gives 2 i cells and a $i+1$ cell, which we call $e \times 0$, $e \times 1$ and $e \times I$. This gives a map $K_i(X') \to K_{i+1}(X' \times I)$, which extends to $C_i(X') \to C_{i+1}(X' \times I)$.

Note that $g \colon (X \times I)_n \to Y$. Then we consider $\theta_g \in C^{n+1}(X \times I, \pi_n(Y))$. The above map induces a map $C^{n+1}(X \times I, \pi_n(Y)) \to C^n(X, \pi_n(Y))$. We call δ the image of θ_g. That is, $\delta(e) = \theta_g(e \times I)$. Let's see what is $d\delta$. We have $\partial(e \times I) = (\partial e) \times I + (-1)^n(e \times 1 - e \times 0)$. Then

$$(d\delta)(e) = \delta(\partial e) = \theta_g(\partial e \times I) = d\theta_g(e \times I) - (-1)^n(\theta_g(e \times 1) - \theta_g(e \times 0)) = (-1)^{n+1}(\theta_{f_1} - \theta_{f_2})(e).$$

So $d\delta = (-1)^{n+1}(\theta_{f_1} - \theta_{f_2})$. In particular, $[\theta_{f_1}] = [\theta_{f_2}] \in H^{n+1}(X, A, \pi_n(Y))$.

Moreover, using homotopy extension we can prove that for $f_0 \colon X_n \to Y$ and $\delta \in C^n(X, A, \pi_n(Y))$, then there is $f_1 \colon X_n \to Y$ homotopic to f_0 on X_{n-1} such that $d\delta = (-1)^{n+1}(\theta_{f_1} - \theta_{f_0})$.

Now if $[\theta_{f_0}] = 0$, we can use the above to produce $f_1 \colon X_n \to Y$ with $\theta_{f_1} = 0$, so f_1 extends to X_{n+1}. □

Corollary 12.4. *If $H^{n+1}(X, A, \pi_n(Y)) = 0$, then any map $A \to Y$ extends to X. Moreover, if $H^n(X, A, \pi_n(Y)) = 0$, then the extension is unique.*

13. 11/03/2020

13.1. **Vector bundles.**

Definition 13.1. A vector bundle over $\mathbb{R}$ is a fiber bundle $p \colon E \to B$ with a addition map $E \times_B E \to E$ over B, together with a multiplication by scalar map $\mathbb{R} \times E \to E$ over B such that: (i) the data give the fibers a finite dimensional vector space structure, (ii) the trivializations of the fiber bundle can be chosen to be fiber-wise linear isomorphisms.

We will denote the trivial bundle $B \times \mathbb{R}^n$ by $n\varepsilon_B$.

Example 13.2. Tangent bundle of a smooth manifold. $\mathbb{R}P^n$ has the tautological bundle, usually denoted λ, same thing about the Grassmanian.

For $(E, B), (E', B')$, can construct $(E \times E', B \times B')$. If $B = B'$, we get $E \oplus E' := E \times E' \times_{B \times B} B$ via $B \xrightarrow{\Delta} B \times B$ the *Whitney sum*. Can do the same thing for the tensor product (externally and internally). Also can take $\mathrm{Hom}_B(E, E')$.

We can consider $\mathrm{Vect}_n\colon \mathrm{Top}^{op} \to \mathrm{Set}$.

Theorem 13.3. *It is a homotopy functor, and is representable.*

We can consider *metrics*, by which we mean a fiberwise inner product (always exist it there is a countable trivializing cover). In particular, any short exact sequence of vector bundles split by choosing a metric.

For (E, B), consider $P_b = \{f\colon \mathbb{R}^n \xrightarrow{\sim} E_b\}$, and $P = \coprod_b P_b \to B$. This admits a topology, since for $n\varepsilon_B$ gives $P = B \times \mathrm{GL}_n(\mathbb{R})$. This is called the *principal bundle* of (E, B). It has a section exactly when (E, B) is trivial. It also has an action of $\mathrm{GL}_n(\mathbb{R})$ that is free, and $P/\mathrm{GL}_n(\mathbb{R}) = B$.

This is the definition of a principal G-bundle: P with a G action that is free, and $P \to P/G$ a fiber bundle. We can produce a n-dim vector bundle out of this by taking $P \times \mathbb{R}^n/((\varphi g, v) \sim \varphi, gv))$. This is the *Borel construction*. This defined a functor $\mathrm{Bun}_G\colon \mathrm{Top} \to \mathrm{Set}$.

Theorem 13.4. *This is a homotopy functor, and representable.*

14. 30/03/2020

14.1. **Spectral sequences.** Consider a fiber bundle $p\colon E \to B$, with homotopy fiber F. How can we relate $H_*(E), H_*(B), H_*(F)$?

Let us assume that B is a CW complex. Then we have a filtration $Sk_n B$, which pullback to a filtration $F_k E = p^{-1}(Sk_k B)$.

This filtration induces a filtration on the singular chain complex: $F_k S_*(E) = S_*(F_k E) \subseteq S_*(E)$. One example of this is cellular homology when p is the identity.

Today: will see some examples. Consider $p\colon B \times F \to B$. If we assume that our coefficients are fields, then we have the Künneth isomorphism and so $H_*(B \times F) = H_*(B) \otimes H_*(F)$. In the case of the Hopf fibration $S^3 \to S^2$, we can see that we have a differential from $(2, 0)$ to $(0, 1)$. The same thing happens with the other Hopf map $S^7 \to S^4$ but with a longer differential.

15. 03/04/2020

15.1. **Serre Spectral sequence.** For $p\colon E \to B$ a fibration, let $F_r E = p^{-1} Sk_r(B)$.

This gives us $E^2_{s,t} = H_s(B, H_t(F))$, where $H_t(F)$ is a local system.

Theorem 15.1. *If $E \to B$ is a Serre fibration, there is a natural first quadrant spectral sequence*

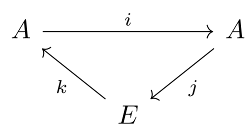
$$E^2_{s,t} = H_s(B, H_t(p^{-1}(*)) \Longrightarrow H_{s+t}(E).$$

Example 15.2. Let's compute $H_*(\Omega S^n)$ for $n > 1$. We have the fibration $PS^n \to S^n$ with fiber ΩS^n. Then $E^2_{s,t} = H_s(S^n) \otimes H_t(\Omega S^n)$. Since we know the answer, we can compute $H_k(\Omega S^n)$ to be $\mathbb{Z}$ when $n-1 \mid k \geq 0$, and 0 otherwise.

16. 06/04/2020

16.1. **Exact couples.**

Definition 16.1. An exact couple is a diagram of abelian groups



which is exact.

In this situation, $d\colon E \to A \to E$ is a differential.

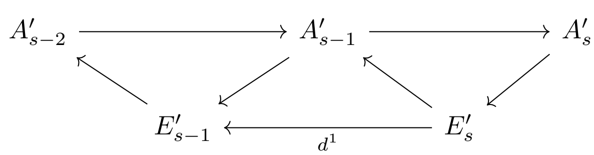
The derived couple is another exact couple A', E' with $A' = \text{im } A$, and $E' = H(E, d)$. i' is just i. j' is $j'(ia) = j(a)$, which is well defined. For k', let $e \in E$ be a cycle, so $jke = 0$, then $ke \in A'$, so define $k'(e) = ke$.

We may do this repeatedly, getting $A^{(r)}$ and $E^{(r)}$.

We give $\mathbb{Z} \times \mathbb{Z}$ grading to an exact couple. We say it is of type r if $\deg i = (1, -1)$, $\deg j = (0, 0)$ and $\deg k = (-r, r-1)$. Then $\deg d = (-r, r-1)$.

For a filtered complex $F_\bullet C_\bullet$, we get an exact couple $A'_{s,t} = H_{s+t}(F_s C_\bullet)$ and $E'_{s,t} = H_{s+t}(\text{gr}_s C_\bullet)$.

Given a type 1 exact couple A', E', write $A^r := (A')^{(r-1)}$ and $E^r := (E')^{(r-1)}$. We can put the exact sequences side by side with the s degree, and the differential show up



and now a diagram chasing from $x \in E'_s$ with $d^1 x = 0$, we can naturally find $x_1 \in A'_{s-2}$, and we define $d^2(x) := j(x_1)$. If $d^2(x) = 0$, then can do the same thing to give $d^3(x)$. If all the differentials are 0, we would hope that x lifts to A'_s.

Example 16.2. Postnikov tower: Y pointed path connected, have the tower of fibrations $Y \to \cdots \to Y[3] \to Y[2] \to Y[1]$, with homotopy fibers being $K(\pi_k(Y), k)$. Applying π_* give us long exact sequences. We may also apply $\pi_*((-)_*^X)$ for some pointed space X (still a tower of fibrations). E^1 will be the new fibers $K(\pi_k(Y),k)_*^X$, whose π_n is:

$$\pi_n(K(\pi_s(Y),s)_*^X) = [S^n \wedge X, K(\pi_s(Y),s)]_* = [X, K(\pi_s(Y), s-n)]_* = \overline{H}^{s-n}(X, \pi_s(Y)).$$

Have to be careful: convergence issues, some of π_n are not abelian. This works stably.

Example 16.3 (Atiyah–Hirzebruck–Serre)**.** $R_*(-)$ a generalized homotopy theory. Exact sequence of pairs give us an exact couple so that we have a spectral sequence of a fibration $E^2_{s,t} = H_s(B, R_t(p^{-1}(-))) \Longrightarrow R_{s+t}(E)$. Though now we do not necessarily have a filtration, as we don't have chain complexes. If p is trivial: $H_s(B, R_t(*)) \Longrightarrow R_{s+t}(B)$.

17. 10/04/2020

17.1. **Hurewicz theorem.** Suppose $E \to B$ is a simple system (i.e. the local system is trivial). Assume $\overline{H}_s(B) = \overline{H}_t(F) = 0$ for $s < p, t < q$. For a while, all differentials are transgressions:

$$E^\infty_{s,0} \to H_r(B) \xrightarrow{d^s} H_{s-1}(F) \to E^\infty_{0,s-1} \to 0$$

for $s < p+q$ and

$$0 \to E^\infty_{0,n} \to H_n(E) \to E^\infty_{n,0}.$$

for $n < p+q$.

We can put these together

$$H_{p+q-1}(F) \to H_{p+q-1}(E) \to H_{p+q-1}(B) \to H_{p+q-2}(F) \to \cdots$$

It looks like the homotopy sequence! And in fact we have a map from the homotopy one to this one by the definition of the transgression.

Corollary 17.1. *Let X be $(n-1)$-connected for $n \geq 2$. Then $\overline{H}_i(X) \xrightarrow{\sim} \overline{H}_{i-1}(\Omega X)$ for $i \leq 2n-2$.*

Theorem 17.2 (Hurewicz)**.** *If X is $(n-1)$-connected for $n \geq 2$, $\pi_n(X) \simeq H_n(X, \mathbb{Z})$.*

Proof. Induction on n. □

17.2. **Relativity.** For a pair of spaces (B, A), say B path-connected, and local systems trivial on

$$\begin{array}{ccc} E|_A & \longrightarrow & E \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Then we get

$$H_s(B, A; H_t(F)) \Longrightarrow H_{s+t}(E, E|_A).$$

Also can consider making it relative along the fiber than the base.

Theorem 17.3. *Assume $A \subseteq X$ both are simply connected and $n \geq 3$. Assume $\pi_i(X, A) = 0$ for $1 \leq i < n$. Then $H_i(X, A) = 0$ for $i < n$ and $h\colon \pi_n(X, A) \simeq H_n(X, A)$.*

Proof. First check that $H_2(X, A) = 0$ by comparing the Hurewicz maps. Using the regular Hurewicz theorem.

Now do induction on n. If F is the homotopy fiber of $A \to X$, we have $\pi_q(X, A) = \pi_{q-1}(F)$. Now study the Serre spectral sequence

$$E^2_{s,t} = H_s(X, A, H_t(\Omega X)) \Longrightarrow H_{s+t}(PX, F) \simeq \pi_{s+t-1}(F) \simeq \pi_{s+t}(X, A).$$

By the universal coefficient theorem, we have $E^2_{s,t} = 0$ for $s < n$.

This means it is an isomorphism in degree n: $H_n(X, A, H_0(\Omega X)) \simeq \pi_n(X, A)$ as we wanted. □

18. 13/04/2020

Note about last class: relative Hurewicz implies Whitehead:

Theorem 18.1 (Whitehead)**.** *Any weak equivalence induces homotopy in homology.*

If X and Y are simply connected (or even simple), then the converse is true: isomorphism in homology implies weak equivalence.

Combined with Whitedead's little theorem, this gives a homotopy equivalence is X, Y are CW complexes.

18.1. **Cohomology Serre Spectral Sequence.** To filter the cohomology, we filter them by kernels $F_{-s}S^k(X) = \ker(S^*(X) \to S^*(F_{s-1}X)$. Define $F^s = F_{-s}$.

This gives a spectral sequence $d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$, with $E_0^{s,t} = \mathrm{gr}^s C^{s+t}$.

We also have a first-quadrant condition: $F^0C^\bullet = C^\bullet$, $\bigcap F^sC^\bullet = 0$ and $H^n(\mathrm{gr}^s C^\bullet) = 0$ for $n < s$. In this case, $\mathrm{gr}^s H^{s+t}(C^\bullet) = E_\infty^{s,t}$.

Theorem 18.2 (Serre spectral sequence)**.** $E_2^{s,t} = H^s(B, H^t(p^{-1}(-))) \Longrightarrow H^{s+t}(E)$.

18.2. **Products.** Now let R be a commutative ring of coefficients.

Then we have $H^*(p^{-1}(-)) \colon \pi_1(B)^{op} \to \{\text{commutative graded } R \text{ algebras}\}$. Then $H^*(B, H^t(p^{-1}(-)))$ is also a bigraded commutative R-algebra: $yx = (-1)^{|x||y|}xy$ where $|x|$ is the total degree of x.

Then the cohomology spectral sequence is multiplicative, in the following sense (more natural using Dress's construction of spectral sequences):

- Each $E_r^{s,t}$ is commutative bigraded algebra.
- $d_r(x, y) = (d_r x)y + (-1)^{|x|} x d_r y$.
- $E_{r+1} = H(E_r)$ as algebras.
- $F^s H^\bullet(E) \cdot F^{s'} H^\bullet(E) \subseteq F^{s+s'} H^\bullet(E)$ and $\mathrm{gr}^\bullet(H^\bullet(E)) \simeq E_\infty^{\bullet,\bullet}$ as algebras.
- $E_2^{\bullet,\bullet} = H^\bullet(B, H^\bullet(p^{-1}(-)))$ as algebras.

What happens in the Gysin sequence: $E \to B$ fibration by homology spheres, oriented: $H^*(p^{-1}(-), R) \simeq H^*(S^{n-1}, R)$.

$$\cdots \to H^{s-n}(B) \to H^s(B) \to H^s(E) \to H^{s-n+1}(B) \to \cdots$$

Consider $\sigma \in H^0(B, H^{n-1}(p^{-1}(-))$ such that $\left\langle \mathrm{res}_b(\sigma), [p^{-1}(-)] \right\rangle = 1$ (canonical given the orientation). Let $e = d_n \sigma \in H^n(B)$. This is called the Euler class. It is a characteristic class: it is natural with respect to pullbacks.

Now the map $H^{s-n}(B) \to H^s(B)$ is given by $x \mapsto d_n(x \cdot \sigma) = (d_n \cdot x)\sigma \pm x \cdot d_n \sigma = \pm e \cdot x$.

If a spherical bundle $E \to B$ has a section, then $e = 0$.

Theorem 18.3. *Considering the tanget bundle $\tau \colon TM \to M$ for a R-oriented closed manifold M, then $\langle e_\tau, [M] \rangle = \chi(M)$.*

The third map is like integrating over the fibers.

Example 18.4. $H^*(\Omega S^n)$ as a ring. Let x be a generator of $H^{n-1}(\Omega S^n)$ with $d_n x = \iota_n$.

Then if n is odd we have $d_n(x^2) = 2x\iota_n$, but d_n is an isomorphism, so there is γ_2 such that $2\gamma_s = x^2$. The same thing says that there is γ_n such that $n!\gamma_n = x^n$. So $H^*(\Omega S^n)$ is a divided power algebra $\Gamma[x]$.

If n is even, then $x^2 = 0$, so choose y such that $d_n y = \iota_n x$. Then $H^*(\Omega S^n) = R[x]/x^2 \otimes \Gamma[y]$.

19. 15/04/2020

19.1. **Serre classes.** Let X be a simply connected space, and $\overline{H}_*(X, \mathbb{Q}) = 0$. Then is $\pi_*(X) \otimes \mathbb{Q} = 0$?

If $\overline{H}_*(X)$ is p-torsion, is $\pi_*(X)$ also p torsion?

If $H_*(X)$ are finitely generated, are $\pi_*(X)$ also finitely generated?

How about in-a-range versions?

Idea: Set up a class of "negligable" groups.

Definition 19.1. A class $\underline{C}$ of abelian groups is a Serre class if $0 \in \underline{C}$, and if $0 \to A \to B \to C \to 0$, then $B \in \underline{C} \iff A, C \in \underline{C}$.

It is automatically closed under isomorphism, subgroups and quotients.

Proposition 19.2. *If $A \to B \to C$ is exact and $A, C \in \underline{C}$, then $B \in \underline{C}$.*

Example 19.3. Trivial groups, finite abelian groups, finitely generated abelian groups, torsion abelian groups, p-torsion abelian groups, p-divisible abelian groups.

Intersection of Serre classes is a Serre class.

We work modulo a Serre class. For example, $f\colon A \to B$ is an isomorphism modulo C_{tor} if and only if $f\colon A \otimes \mathbb{Q} \xrightarrow{\sim} B \otimes \mathbb{Q}$.

Lemma 19.4. *Let $\underline{C}$ be a Serre class. Then monomorphisms, epimorphisms and isomorphisms mod $\underline{C}$ are closed under composition.*

Moreover, isomorphisms modulo C satisfy the two out of three property.

Proof. Use the exact sequence of composition. □

Suppose $C_\bullet$ is a chain complex. If $C_\bullet$ is in $\underline{C}$, then so is $H_\bullet(C_\bullet)$.

If $F_\bullet A$ is a filtered abelian group and the filtration is finite, then if $\mathrm{gr}_s A$ is in $\underline{C}$, then so is A.

So if we have a first quadrant spectral sequence which is eventually 0 mod $\underline{C}$, then the converging object is also.

Definition 19.5. A Serre class $\underline{C}$ is a Serre ring if $A, B \in \underline{C}$ implies $A \otimes B, \mathrm{Tor}(A, B) \in \underline{C}$.

It is a Serre ideal if the same conclusion holds when only A or B is in $\underline{C}$.

Theorem 19.6 (Mod $\underline{C}$ Vietoris–Bagle)**.** *Let $p\colon E \to B$ is a fibration with B path-connected with path-connected fiber F. Assume $\pi_1(B)$ acts trivially on $H_\bullet(F)$. Let $\underline{C}$ be a Serre ideal, and $H_t(F) \in \underline{C}$ for $t > 0$. Then $p_*\colon H_\bullet(E) \to H_\bullet(B)$ is an isomorphism mod $\underline{C}$.*

Proof. Follows from the Serre spectral sequence since $E^2_{s,t} \in \underline{C}$ for $t > 0$. So $E^\infty_{s,t} = 0$ for $t > 0$. So the edge homomorphisms (the maps we want) are isomorphisms mod $\underline{C}$. □

Theorem 19.7. *Let $p\colon E \to B$ is a fibration, B $\pi_1(B)$ acts trivially as before and F path connected. Let $\underline{C}$ be a Serre ring. Assume that $H_s(B) \in \underline{C}$ for $0 < s < n$ and $H_t(F) \in \underline{C}$ for $0 < t < n-1$. Then $p_*\colon H_i(E, F) \to H_i(B, *) = \overline{H}_i(B, *)$ is a mod $\underline{C}$ isomorphism for $i \leq n$.*

Proof. Use the relative Serre spectral sequence

$$E^2_{s,t} = \overline{H}_s(B, H_t(F)) \Longrightarrow H_{s+t}(E, F).$$

□

Theorem 19.8 (Mod $\underline{C}$ Hurewicz theorem)**.** *Let $\underline{C}$ be a Serre ring such that*

$$A \in \underline{C} \implies H_q(K(A, 1)) \in \underline{C} \text{ for } q \geq 1.$$

Let X be simply connected, $n \geq 2$. Then we have

$$\pi_q(X) \in \underline{C} \text{ for } q < n \iff \overline{H}_q(X) \in \underline{C} \text{ for } q < n,$$

and in such case, $\pi_n(X) \to H_n(X)$ is an isomorphism mod $\underline{C}$.

Remark 19.9. It is not hard to prove that the above extra condition is true, for instance, for $\underline{C}_{tor}$: it boils down to computing the homology of $K(\mathbb{Z}/n, 1)$. This can be done by considering $B\mathbb{Z}/n \to BS^1$ with fiber $S^1/(\mathbb{Z}/n) \simeq S^1$.

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Consider the Serre class C_p the torsion abelian subgroups without p-torsion.

Then $A \in C_p$ if and only if $A \otimes \mathbb{Z}_{(p)} = 0$.

Lemma 20.1. *Let X, Y such that $H_*(-, \mathbb{Z}_{(p)})$ is of finite type, and $X \to Y$ an isomorphism on $H_*(-, \mathbb{F}_p)$. Then $H_*(X) \to H_*(Y)$ is an isomorphism modulo C_p.*

Definition 20.2. A Serre class C is acyclic if $A \in C$ implies $H_q(K(A,1)) \in C$ for all $q \geq 1$.

Theorem 20.3. *C an acyclic Serre ring, X simply connected, $n \geq 2$. Then $\pi_q(X) \in C$ for $q < n$ if and only if $\overline{H}_q(X) \in C$ for $q < n$. In such case, $\pi_n(X) \to H_n(X)$ is an isomorphism mod C.*

Proof. As before, by induction

$$\begin{array}{ccccc} \pi_n(X) & \xleftarrow[\sim]{} & \pi_n(PX, \Omega X) & \xrightarrow{\sim} & \pi_{n-1}(\Omega X) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{H}_n(X) & \longleftarrow & H_n(PX, \Omega X) & \xrightarrow{\simeq} & H_{n-1}(\Omega X) \end{array}$$

The difference is that for ΩX it is not easy to apply the induction hypothesis.

We get rid of $\pi_2(X)$ by considering $Y = X[3,\infty) \to X$ with homotopy fiber $K := K(\pi_2(X),1)$. By acyclicity, $H_i(Y,*) \to H_i(Y,K)$ is a mod C isomorphism, and also $H_i(Y,*) \to H_i(X,*)$, since we can use the spectral sequence. This implies $H_i(Y,K) \to H_i(X,*)$ is a mod C isomorphism. This proves that both maps that we want in the diagram are isomorphisms mod C. □

Corollary 20.4. *Let X simply connected, p a prime. Then*

$$\pi_i(X) \otimes \mathbb{Z}_{(p)} = 0 \text{ for } i < n \iff \overline{H}_i(X, \mathbb{Z}_{(p)} = 0 \text{ for } i < n$$

and in this case $\pi_n(X) \otimes \mathbb{Z}_{(p)} \xrightarrow{\sim} H_n(X, \mathbb{Z}_{(p)})$.

Theorem 20.5 (Relative mod C Hurewicz)**.** *Let C be an acyclic Serre ideal. Let (X,A) be a pair both simply connected, and $n \geq 2$. Then*

$$\pi_i(X,A) \in C \text{ for } i < n \iff H_i(X,A) \in C \text{ for } i < n$$

and in such case $\pi_n(X,A) \to H_n(X,A)$ is a C isomorphism.

Proof. Let F be the homotopy fiber

$$\begin{array}{ccccc} \pi_n(X,A) & \xleftarrow[\sim]{} & \pi_n(PX, F) & \xrightarrow{\sim} & \pi_{n-1}(F) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{H}_n(X,A) & \longleftarrow & H_n(PX, F) & \xrightarrow{\simeq} & H_{n-1}(\Omega F) \end{array}$$

The right one is fine by the absolute Hurewicz. For the other one, look at the relative Serre spectral sequence. □

Theorem 20.6 (Mod C Whitehead)**.** *Let C be an acyclic Serre ideal. Let $F \colon C \to Y$ a map of simply connected spaces. Then the following are equivalent: (i) $\pi_i(X) \to \pi_i(Y)$ is an isomorphism* *mod C for $i < n$ and C-epi for $i = n$, (ii) $H_i(X) \to H_i(Y)$ is a C-iso for $i < n$ and C-epi for $i = n$.*

Corollary 20.7. *Let $f\colon X \to Y$ simply connected and $H_*(-, \mathbb{Z}_{(p)})$ of finite type. If $H_*(f, \mathbb{F}_p)$ is iso, then*

$$\pi_*(X) \otimes \mathbb{Z}_{(p)} \xrightarrow{\sim} \pi_*(Y) \otimes \mathbb{Z}_{(p)}.$$

Example 20.8. Let us compute some $H^*(K(A, n), \mathbb{Q})$.

If A is a torsion group, using C_{tor} we get $\overline{H}_* = 0$.

If $A = \mathbb{Z}$, then $K(\mathbb{Z}, 1) = S^1$, and $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. We can compute for $n = 2$ instead by using the fibration $K(\mathbb{Z}, 1) \to PK(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$. This generalizes for all n using the multiplicativity of the spectral sequence. We get $H^*(K(\mathbb{Z}, n), \mathbb{Q})$ is $\mathbb{Q}[\iota_n]$ is n is even, and $E[\iota)n]$ is n is odd (exterior algebra).

Example 20.9. Let us compute $\pi_*(S^n) \otimes \mathbb{Q}$. First note that $C_{f.g.}$ implies that $\pi_* S^n$ are finitely generated. Let $S^n \to K(\mathbb{Z}, n)$ be a generator (of $\pi_n(K(\mathbb{Z}, n))$ and $H^n(S^n)$). For n odd, this is a $\mathbb{Q}$-iso in H_*, hence in $\pi_* \otimes \mathbb{Q}$.

For n even, let F be the homotopy fiber of $S^n \to K(\mathbb{Z}, n)$. Analyzing the spectral sequence, we get that H^*F is exterior with a class in dimension $2n - 1$. So there is a map $S^{2n-1} \to F$ with is a Q-isomorphism.

So

$$\pi_i(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = n, \\ \mathbb{Q} & \text{if } i = 2n - 1 \text{ and } 2 \mid n, \\ 0 & \text{otherwise.} \end{cases}$$