18.725: ALGEBRAIC GEOMETRY, FALL 2019

CHENYANG XU, NOTES BY MURILO ZANARELLA

Contents

Problem sets

Website

Book

1. 05/09/2019

We will begin the class with chapter 1, which is the classical language, and them go to chapter 2, which is the language by Grothendieck.

Let $k = \overline{k}$ be an algebraically closed field.

Definition 1.1. Denote $\mathbb{A}^n$ the affine space over k, and for a $f \in k[x_1, \ldots, x_n]$, consider the set

$$Z(f) := \{(x_1, \ldots, x_n) \colon f(x_1, \ldots, x_n) = 0\},$$

and analogously for an ideal $I \subseteq k[x_1, \ldots, x_n]$, we consider $Z(I)$, and these are called *algebraic sets*. We endow $\mathbb{A}^n$ with the *Zariski topology*, in which the closed sets are the algebraic sets.

Remark 1.2. Since $k[x_1, \ldots, x_n]$ is Noetherian, all algebraic sets are the zero set of finitely many polynomials.

Example 1.3. $\mathbb{A}^1$ has the cofinite topology.

Definition 1.4. An algebraic set Y is called irreducible if Y cannot be written as a union of proper algebraic sets.

Definition 1.5. An *affine variety* is an irreducible algebraic set of one of $\mathbb{A}^n$. A *quasi-affine variety* is an open set of an affine variety.

Now start with a subset $Y \subseteq \mathbb{A}^n$, and we consider

$$I(Y) := \{f \in k[x_1, \ldots, x_n] \colon f(Y) = 0\}.$$

Proposition 1.6. *Z and I satisfy the following properties:*

1. *Z and I are inclusion-reserving,*
2. *$I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$,*
3. *if $\mathfrak{a}$ is an ideal of $k[x_1, \ldots, x_n]$, then $I(Z(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$,*
4. *for any Y, $Z(I(Y)) = \overline{Y}$.*

Proof. First two are easy, (3) is Nullstelenzatz, and for (4), note that $Y \subseteq Z(I(Y))$ implies $\overline{Y} \subseteq Z(I(Y))$, and if we have any other $W = Z(\mathfrak{a})$, then $W \supseteq Y$ implies by (1) that $\mathfrak{a} \subseteq I(Z(\mathfrak{a})) \subseteq I(Y)$, and so again by (1), we have $W = Z(\mathfrak{a}) \supseteq Z(I(Y))$. $\square$

Corollary 1.7. *There exist a one to one correspondence*

$$\{\textit{algebraic subset of } \mathbb{A}^n\} \longleftrightarrow \{\textit{radical ideals of } k[x_1, \ldots, x_n]\}$$

given by Z and I. Moreover, under this correspondence we have

$$\{\textit{subvarieties of } \mathbb{A}^n\} \longleftrightarrow \{\textit{prime ideals of } k[x_1, \ldots, x_n]\}.$$

Example 1.8. $\mathbb{A}^n$ is irreducible, $Z(f)$ for an irreducible polynomial f is irreducible.

Definition 1.9. For an algebraic set Y, let its *coordinate ring* be

$$k[x_1, \ldots, x_n]/I(Y).$$

Definition 1.10. A topological space A is *Noetherian* is any descending chain of closed sets stabilize.

Proposition 1.11. *If X is a Noetherian topological space, then any closed subspace Y can be decomposed into irreducibles subspaces in a unique way if one does not allow redundancies.*

2. 10/09/2019

Example 2.1. Let $I = (x^2 - yz, xz - x) \subseteq k[x, y, z]$. Then $Z(I) = Z(x^2 - yz) \cap Z(xz - x) = Z(x^2 - yz) \cap (Z(z - 1) \cup Z(x)) = (Z(x^2 - yz) \cap Z(z - 1)) \cup (Z(x^2 - yz) \cap Z(x))$, which is $Z(x^2 - y, z - 1) \cup Z(x, y) \cup Z(x, z)$, and has three components.

Definition 2.2. The *dimension* of a Noetherian topological space if the largest d such that there exist irreducible closed subsets

$$Z_0 \subset Z_1 \subset \cdots \subset Z_d \subseteq Z.$$

Definition 2.3. For a quasi-affine variety (or an algebraic set) X, we define $\dim X$ to the the dimension of the Zariski topology associated to X.

Proposition 2.4. *If Y is an affine algebraic set, then $\dim Y = \dim A(Y)$, the Krull dimension of $A(Y)$.*

Theorem 2.5. *Let k be a field and B a finitely generated k-algebra with B integrally closed. Then*

(a) $\dim B$ *is the transcendence degree of* $\mathrm{Frac} B/k$, *(b) for any prime ideal* $\mathfrak{p}$,

$$\dim B = \operatorname{height} \mathfrak{p} + \dim B/\mathfrak{p}.$$

Proposition 2.6. *If* Y *is quasi-affine, then* $\dim Y = \dim \overline{Y}$.

Proof. It is clear that $\dim Y \leq \dim \overline{Y}$: if $Z_i \subset Z_{i+1}$ are closed in Y, then their closure are still distinct.

Now consider a chain of maximal length

$$\{P\} = Z_0 \subset \cdots \subset Z_n \subseteq Y.$$

Since $\dim \overline{Y} = \operatorname{height}_{A(\overline{Y})} \mathfrak{m}/I(\overline{Y}) + \dim B/\mathfrak{m}$ where $\mathfrak{m} = I(Z_0) \supseteq I(\overline{Y})$ is a maximal ideal, we have $\dim \overline{Y} \geq (\dim Y - 1) + 1 = \dim Y$. □

Theorem 2.7. *Let* A *be a Noetherian ring, and* $f \in A$ *neither a unit nor a zero divisor. The every minimal prime* $\mathfrak{p}$ *containing* f *has height* 1.

Proposition 2.8. *An integal domain* A *is a UFD if and only if any height 1 prime ideal is principal.*

Theorem 2.9. *An addine variety* $Z \subseteq \mathbb{A}^n$ *is of dimension* $N - 1$ *id and only if* $Z = Z(f)$ *for some* f *irreducible.*

Proof. Assume first $\dim Z = n - 1$. If $Z = Z(\mathfrak{p})$, then $n = \operatorname{height} \mathfrak{p} + \dim Z$, and so $\mathfrak{p}$ is height 1, and hence principal.

Conversely, use the theorem above to conclude that the dimension is $n - 1$. □

2.1. **Projective varieties.**

Definition 2.10. The *projective space* $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/\sim$ where $x \sim \lambda x$ for $\lambda \in k^\times$. We consider the ring $S = \sum_{i \geq 0} S_i$ where S_i is the set of degree i homogeneous polynomials. An ideal $\mathfrak{a}$ is a *homogeneous* ideal if $\mathfrak{a} = \sum_{i \geq 0} S_i \cap \mathfrak{a}$.

Definition 2.11. $Y \subseteq \mathbb{P}^n$ is an *algebraic set* it it is $Z(T)$ there T is a set of homogeneous polynomials, and we define the Zariski topology on $\mathbb{P}^n$ by taking the closed sets to be the algebraic sets.

Definition 2.12. A *projective variety* is an irreducible algebraic set of $\mathbb{P}^n$. A *quasi-projective variety* is an open set of a projective variety.

Definition 2.13. For an algebraic set Y we let $I(Y)$ be the homogeneous ideal of homogenous polynomials that vanish on Y. We denote $A(Y) = S/I(Y)$ the *homogeneous coordinate ring.*

Proposition 2.14 (Homogeneous Nullstellensatz)**.** *Let $\mathfrak{a} \subseteq S$ be a homogeneous ideal. If $f \in S$ with positive degree, then $f(Z(\mathfrak{a})) = \{0\}$ if and only if $f^p \in \mathfrak{a}$ for some p.*

Proof. Consider the ideal $\mathfrak{a}$ inside of $\mathbb{A}^{n+1}$. Since $\deg f > 0$, it also vanishes at $0 \in \mathbb{A}^{n+1}$. By the usual Nullstellensatz, we have $f^p \in \mathfrak{a}$. □

Definition 2.15. The *hyperplanes* are $H_i = Z(x_i)$, and $S \setminus H_i$ is a copy of $\mathbb{A}^n$.

Proposition 2.16. *$\varphi \colon \mathbb{A}^n \to \mathbb{P}^n \setminus H_0$ by $(y_1, \dots, t_n) \mapsto (1, y_1, \dots, y_n)$ is a homeomorphism.*

3. 12/09/2019

Proof. For $f \in S^h$, we let $\alpha(f) = f(1, y_1, \dots, y_n) \in A := k[y_1, \dots, y_n]$. For $g \in A$, we let $\beta(g) = x_0^{\deg g} g(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$. We want to see these send closed sets to closed sets.

If Y is closed on $\mathbb{P}^n \setminus H_0$, we have $\overline{Y} = Z(T)$ and $\overline{Y} \cap (\mathbb{P}^n \setminus H_0) = Y$. Then $Z(\alpha(T)) = \varphi(Y)$.

Now let $W = Z(T')$. Then $\varphi^{-1}(W) = Z(\beta(T')) \setminus H_0$. □

Example 3.1. Consider the map $\mathbb{P}^n \to \mathbb{P}^N$ taking $(x_0, \dots, x_n) \mapsto \left(t_I := \prod_{i \in I} x_i\right)$ for I multisets of d elements, so $N = \binom{n+d-1}{n}$. The image will be a projective variety, given by equations

$$T := \left\{ \prod_{i=1}^k y_{I_i} = \prod_{j=1}^k y_{J_j} \text{ if } \bigcup_i I_i = \bigcup_j J_j \right\}.$$

We know that the image of the map above lies inside $Z(T)$, and we want to prove it maps surjectively. Indeed, let $p \in Z(T)$. Consider $y_{(i_1, \dots, i_d)}$ such that its coordinate is nonzero. We then consider $x_i = \frac{y_{(i, i_2, \dots, i_d)}}{y_{(i_1, i_2, \dots, i_d)}}$. Then $x_{k_1} \cdots x_{k_d} = \frac{y_{(k_1, \dots, k_d)}}{y_{(i_1, \dots, i_1)}}$. We can see the denominator is nonzero because $y_{(i_1, \dots, i_d)}^d = \prod_{j=1}^d y_{(i_j, \dots, i_j)}$. Then $(x_0, \dots, x_n)$ maps to p.

3.1. Morphisms.

Definition 3.2. Let Y be a quasi-affine variety of $\mathbb{A}^n$. A *regular function* f at $p \in Y$ is such that there is a neighborhood U with $p \in U \subseteq Y$ such that there are $g, h \in A$ such that $f = \frac{g}{h}$ in U.

Lemma 3.3. *A regular function $f \colon Y \to \mathbb{A}^1$ is continuous.*

Proof. We only need to show that for $p \in \mathbb{A}^1$ that $f^{-1}(t)$ is closed. To show this, we may show this locally. So choose a neighborhood U of t such that $f = \frac{g}{h}$ at U. Then $g(x) = t \cdot h(x)$ is $f^{-1}(t)$ at U. $\square$

Definition 3.4. Let Y be a quasi-affine projective variety of $\mathbb{P}^n$. A *regular function* f at $p \in Y$ is such that there is a neighborhood U with $p \in U \subseteq Y$ such that there are $g, h \in S^h$ with same degree such that $f = \frac{g}{h}$ in U.

Definition 3.5. A *variety* (over k) is a quasi-projective variety. If X, Y are two varieties, a *morphism* $\varphi \colon X \to Y$ is a map that is continuous, and such that for every open set $V \subseteq Y$ and regular function $V \to \mathbb{A}^1$, we have that $f \circ \varphi \colon \varphi^{-1}(V) \to \mathbb{A}^1$ is regular.

Example 3.6. Consider $\mathbb{A}^1 \to \mathbb{A}^2$ by $t \mapsto (t^2, t^3)$. Then the image is $Z(y^2 - x^3)$. One can check this is a homeomorphism on points. But it is not an isomorphism, since its inverse is not going to be a morphism.

Example 3.7. In characteristic p, look at $\mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. It is also a homeomorphism but is not an isomorphism.

Definition 3.8. If Y is a variety, we consider the objects:

(1) $\mathcal{O}(Y)$ the ring of regular functions,
(2) $\mathcal{O}_p(Y)$ the ring of germs of regular functions at p,
(3) $K(Y)$ the ring of functions that are regular at some point of p, identifying then if they agree locally (this is an equivalence relation).

Theorem 3.9. *If Y is an affine variety, then $\mathcal{O}(Y) = A(Y)$, $\mathcal{O}_p(Y) = A(Y)_{\mathfrak{m}_p}$ where $\mathfrak{m}_p$ is the maximal ideal for p. Moreover, $\dim \mathcal{O}_p(Y) = \dim Y$. Finally, $K(Y) = \operatorname{Frac} A(Y)$.*

Proof. We have a natural injection $A(Y) \hookrightarrow \mathcal{O}(Y)$. Now let $p \in Y$ and $\mathfrak{m}_p$ its maximal ideal. Then $A(Y)_{\mathfrak{m}_p} \hookrightarrow \mathcal{O}_p(Y)$, since they are the localization of integral rings. But it is easy to see it is surjective. Now taking fraction fields, we have $K(Y) = \operatorname{Frac} A(Y)$. Now $A(Y) \hookrightarrow \mathcal{O}(Y) \hookrightarrow \bigcap_p \mathcal{O}_p \hookrightarrow \bigcap_{\mathfrak{m}} A(Y)_{\mathfrak{m}} = \operatorname{Jac}(A(Y))$, which is $A(Y)$ since $A(Y)$ is integral. $\square$

Proposition 3.10. *The map $\mathbb{P}^n \setminus H_i \to \mathbb{A}^n$ is an isomorphism.*

If $\mathfrak{p}$ is a homogeneous prime of S, we consider $S_{(\mathfrak{p})}$ the degree 0 part of $T^{-1}S$ where T is the hoomogeneous polynomials in $S \setminus \mathfrak{p}$.

4. 17/09/2019

Theorem 4.1. *Let $Y \subseteq \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring $S(Y)$. Then $\mathcal{O}(Y) = k$. For all $p \in Y$, let $\mathfrak{m}_p \subseteq S(Y)$ be the ideal generated by all elements in $S(Y)$ with $f(p) = 0$. Then $\mathcal{O}_p \simeq S(Y)_{(\mathfrak{m}_p)}$. Finally, $K(Y) \simeq S(Y)_{(0)}$.*

Proof. Choose $p \in U_0 = \mathbb{P}^n \backslash Z(x_0)$. Then $S(Y)_{(x_0)} \simeq A(Y_0)$ if $Y_0 = Y \cap U_0$. If f' is the corresponding function, we have $\mathcal{O}_p \simeq A(Y_0)_{\mathfrak{m}'_p}$ and we can check that this is $S(Y)_{(\mathfrak{m}_p)}$. For the last part, we have $K(Y) = K(Y_0) = K(A(Y_0)) = S(Y)_{(0)}$.

If $f \in \mathcal{O}(Y)$, then $f|_{U_i} \in S(Y)_{(x_i)}$, that is, there is N_i such that $x_i^{N_i} f \in S(Y)$. Then we have for N large that $S_N f \subseteq S(Y)$, and since the degree of f is 0, this means $S(Y)_N f \subseteq S(Y)_N$, and so $S(Y)_N f^m \subseteq S(Y)_N$ for any m. This implies that $S(Y)[f] \subseteq S(Y)_{x_0}$, and so $S(Y)[f]$ is a finite module over $S(Y)$. So f is integral over $S(Y)$. In particular we can take the degree 0 part of the relation. Hence it has coefficients in $S(Y)_0 = k$, and as k is algebraically closed, this means that $f \in k$. □

Proposition 4.2. *Let X be a variety, and Y an affine variety. Then there is a bijection* $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(A(Y), \mathcal{O}(X))$.

Proof. The map is the pullback, which comes from the definition of a morphism: $\varphi\colon X \to Y$ goes to $h\colon A(Y) \to \mathcal{O}(X)$. We want to construct its inverse. Consider $\overline{x_i} \in A(Y)$ the image of x_i. Write $\xi_i = h(\overline{x_i}) \in \mathcal{O}(X)$. Then we consider $X \to \mathbb{A}^n$ to be $(\xi_1, \dots, \xi_n)$. It suffices now to prove that its image is in Y. For $f \in I(Y)$, we have that $f(\xi_1(p), \dots, \xi_n(p)) = 0$ for all $p \in X$, that is, that $(\xi_1(p), \dots, \xi_n(p)) \in Y$. It remains to check such map is a morphism, which follows from the next lemma. □

Lemma 4.3. *Let X be a variety, and $Y \subseteq \mathbb{A}^n$ an affine variety. Then $\psi\colon X \to Y$ is a morphism if and only if $x_i \circ \psi$ is regular for all i.*

Proof. Let ψ be such that $x_i \circ \psi$ are all regular. To prove ψ is continuous, we need to check that for any regular function and any closed set Z, $f\colon Z \to \mathbb{A}^1$, we have closed preimage. But $\psi \circ f(x_1, \dots, x_n) = f(x_1 \circ \psi, \dots, x_n \circ \psi)$, which is regular.

Now for any regular function at p, there is a neighborhood of $p \in Y$, in which it is given by $\frac{f}{g}$ with $g(p) \neq 0$ for $f, g \in A(X)$. Then $\frac{f}{g} \circ \psi = \frac{f \circ \psi}{g \circ \psi}$ and $f \circ \psi, g \circ \psi$ are regular, with $g \circ \psi(p) \neq 0$, so it is regular. □

Corollary 4.4. *If X and Y are affine, then $X \simeq Y$ if and only if $A(X) \simeq A(Y)$.*

Corollary 4.5. *$X \to A(X)$ is a contravariant equivalence between affine varieties and finitely generated k-algebras which is an integral domain.*

4.1. **Rational maps.**

Lemma 4.6. *Let $\varphi, \psi\colon X \to Y$ two morphisms. If $\varphi = \psi$ in an open nonempty $U \subseteq X$, then $\varphi = \psi$.*

Proof. We may assume that $Y = \mathbb{P}^n$. Considering $\mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$, we may consider (φ, ψ). Then $(\varphi, \psi)|_U \subseteq \Delta$ the diagonal. So its pullback is a closed subsetet of X. Since it also contains U, it must be the entire X. □

Definition 4.7. A *rational map* $X \dashrightarrow Y$ for X, Y varieties is a pair (U, φ_U) such that $U \subseteq X$, $\varphi_U : U \to Y$ modulo the equivalence relation when they agree at the intersection.

Definition 4.8. A *birational map* is $\varphi\colon X \to Y$ a rational map that has a rational map that is its inverse.

5. 19/09/2019

Lemma 5.1. *If Y is a hypersurface in $\mathbb{A}^n$, then $\mathbb{A}^n \setminus Y$ is a hypersurface in $\mathbb{A}^{n+1}$, and in particular is affine.*

Proof. Let $Y = Z(f)$. Define $H \subseteq \mathbb{A}^{n+1}$ to be $Z(f(x_1, \ldots, x_n)x_{n+1} - 1)$. Now we define the natural projection map $H \to \mathbb{A}^n$ onto the first n coordinates, and we can check this is a bijection on points $H \to \mathbb{A}^n \setminus Y$. It is easy to check that $\mathbb{A}^n \setminus Y \to H$ is a morphism, since H is affine and each coordinate is regular. □

Proposition 5.2. *If Y is a variety, then there exist a basis for the topology consisting of affine open sets.*

Proof. Y is covered by quasi-affines, so we may assume Y is quasi-affine. And in fact, we have $\overline{Y} \setminus Y = Z(\mathfrak{a})$ and so there is $f \in \mathfrak{a}$ with $f(p) \neq 0$. By the lemma, $\mathbb{A}^n_f := \mathbb{A}^n \setminus Z(f)$ is affine, and $\mathbb{A}^n_F \cap Y = \mathbb{A}^n_f \cap \overline{Y}$ is closed in $\mathbb{A}^n_f$. This let us conclude that $\mathbb{A}^n_f \cap Y$ is affine, and contains p. □

Definition 5.3. We say a rational map $X \dashrightarrow Y$ is *dominant* if there is $U \subseteq X$ with $\overline{f(U)} = Y$.

Theorem 5.4. *Given X a variety, there is a contravariant equivalence of categories of varieties Y with dominant rational maps from x to Y and field extensions of finitely generated k-algebras of $K(X)$.*

Proof. If f is a regular function on Y, we consider $U \cap f^{-1}(V)$ and choose $U_1 \subseteq U \cap f^{-1}(V)$ affine. Now we can pullback f to U_1 to a regular map $\theta(f)$, so to an element of $\mathcal{O}(U_1) = A(U_1) \hookrightarrow K(A(U_1)) = K(X)$.

Now consider $\theta K(Y) \to K(X)$. We may assume Y is affine, and choose generators $y_1, \dots, y_n$ of $A(Y)$, and we can find an open set $U \subseteq X$ such that $\theta(y_i)$ are regular on U. Then this gives a map $A(Y) \to \mathcal{O}(U)$, which gives a morphism $U \xrightarrow{\varphi} Y$. To see that this is dominant, we use $A(Y) \to \mathcal{O}(U)$ is an injection, as if $f \in Z(\overline{\varphi(U)})$, then f maps to 0 under θ.

It remains to prove that for a field extension K, there is a variety X. But this is simply by finding generators $x_1, \dots, x_n$ of K, and then $k[x_1, \dots, x_n] \subseteq K$, and consider a presentation of such algebra. □

Corollary 5.5. *Let X, Y be varieties. The following are equivalent:*

(1) X and Y are birational.
(2) There are $U \subseteq X$ and $V \subseteq Y$ with $U \simeq V$.
(3) $K(X) \simeq K(Y)$.

Theorem 5.6. *Any variety is birational to a hypersurface.*

Example 5.7 (Blow-up)**.** Consider the algebraic set $X \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ given by $(x_1, \dots, x_n, y_1, \dots, y_n)$ such that $x_i y_j = x_j y_i$ for all i, j. We can show that this is irreducible: for a line L_a through the origin, we have that $\overline{\varphi^{-1}(L_a) \setminus 0}$ where $\varphi \colon X \to \mathbb{A}^n$ is $(a_1 t, \dots, a_n t, a_1, \dots, a_n)$ when t varies. Then $X \setminus \varphi^{-1}(0) \simeq \mathbb{A}^n \setminus 0$ is irreducible, so $X = \overline{X \setminus \varphi^{-1}(0)}$ is also irreducible.

Definition 5.8. If $0 \in Y \subseteq \mathbb{A}^n$ is a variety, we define the blow-up $\mathrm{Bl}_0 Y := \overline{\varphi^{-1}(Y \setminus 0)} \subseteq X$ as above.

Example 5.9. Let $Y = Z(y^2 - x^2(x+1))$. It has a node at 0, but its blowup at 0 gives it a non-intersecting curve: it is given by the equations $y^2 = x^2(x+1)$ and $xv = yu$ for $(u, v) \in \mathbb{P}^1$. This procedure is resolving a singularity.

Definition 5.10. For a variety $Y \subseteq \mathbb{A}^n$ a point $p \in Y$ is *non-singular* or *smooth* if for any generators $f_1, \dots, f_m \subseteq I(Y)$, the Jacobian at p has rank $n - \dim Y$.

6. 24/09/2019

Definition 6.1. A local Noetherian ring R is called *regular* if $\dim R = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2$.

Theorem 6.2. *An affine variety is nonsingular at p if and only if $\mathcal{O}_p$ is regular.*

Proof. We may assume without loss of generality that $p = 0$. For a $f \in A = k[x_1, \dots, x_n]$, we consider $(\partial_1 f(p), \dots, \partial_n f(p)) \in k^n$. This factors through $A/\mathfrak{m}^2$, and so its image is the same as the image of $\mathfrak{m}/\mathfrak{m}^2$. Let the ideal corresponding to Y be $\mathfrak{a}$. Then the rank of the Jacobian matrix is $n_1 = \operatorname{rank}_k(\mathfrak{a} + \mathfrak{m}^2)/\mathfrak{m}^2$ Then the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}_p$ is $\mathfrak{m}/\mathfrak{a}$, and $\operatorname{rank}_k \mathfrak{m}_p/\mathfrak{m}_p^2 = \operatorname{rank}_k \mathfrak{m}/(\mathfrak{a} + \mathfrak{m}^2) = n_2$.

Since $n_1 + n_2 = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 = n$, the claim follows. □

6.1. **Nonsingular curves.**

Definition 6.3. Let K/k be a field extension. A *valuation* $v \colon K^\times \to G$ where G is a totally ordered abelian group satisfy, is a group homomorphism that safisfy $v(x+y) \geq \min(v(x), v(y))$, and $v(k) = 0$.

Proposition 6.4. *For a valuation v on K, $R = \{x \in K \colon v(x) \geq 0\}$ is the valuation ring, and is local with maximal ideal $\{x \in K \colon v(x) > 0\}$.*

Theorem 6.5. *If K is a field, $R \subseteq K$ is a valuation ring for some v if and only if R is maximal element with respect to domination ($(R_2, \mathfrak{m}_2)$ dominates $(R_1, \mathfrak{m}_1)$ iff $R_1 \subseteq R)2$ and $\mathfrak{m}_2 \cap R_1 = \mathfrak{m}_1$). Also, any local ring is dominated by some valuation ring.*

Theorem 6.6. *Let A be a Noetherian local domain of dimension 1. Then the following are equivalent:*

(1) A is a DVR,
(2) A is integrally closed,
(3) A is regular,
(4) $\mathfrak{m}$ is principal.

Definition 6.7. A is a Dedekind domain if A is integrally closed, Noetherian of dimension 1.

Theorem 6.8. *If R is a Dedekind domain, and $L \supseteq K(R)$ is a finite field extension, and R_L is the integral closure of R in L, then R_L is a Dedekind domain.*

Let K/k be a field. We denote C_K the set of all discrete valuation rings of K.

Lemma 6.9. *Let Y a variety. Let $p, q \in Y$. If $\mathcal{O}_q \subseteq \mathcal{O}_p$, then $p = q$.*

Proof. Since $Y, \overline{Y}$ have the same function field, we may assume Y is projective. Now choose a hyperplane that misses p and q. Then $Y - H$ is affine, and the local rings are localizations of $A(Y - H)$, and now the theorem is clear. □

Lemma 6.10. *Let K be a function field of dimension 1. For any $x \in K$, the set $\{R \in C_K : x \notin R\}$ is finite.*

Proof. $x \notin R$ if and only if $y := 1/x \in \mathfrak{m}_R$. If $y \in k$, this is trivial. Otherwise $k[y]$ is a polynomial ring (k is algebraically closed). Now $K/k(y)$ is a finite extension, so we can consider the integral closure B of $k[y]$ in K. Then B is a Dedekind domain. If $y \in R$, then $k[y] \subseteq R$, and so $B \subseteq R$. Then $\mathfrak{m}_R \cap B = \mathfrak{n}$ is a prime ideal, and $B_\mathfrak{n} \subseteq R$ is a DVR. So $B_\mathfrak{n} = R$. So $y \in \mathfrak{m}_R$ if and only if $y \in \mathfrak{n}$. This is the same as the vanishing locus of y containing the point given by $\mathfrak{n}$ on the affine curve given by B. There are finitely many such points. □

Corollary 6.11. *Any DVR in K/k is isomorphic to the local ring of a point at some nonsingular affine curve.*

Proof. The affine curve is the B in the proof above. □

We put the profinite topology on C_K.

Now for any $U \subseteq C_K$, open, we define $\mathcal{O}(U) = \bigcap_{p \in U} R_p$. For $f \in \mathcal{O}(U)$, we define $f(p) = [f] \in R_p/\mathfrak{m}_p \simeq k$. For any $f \in \mathcal{O}(U)$, it vanishes at finitely many points.

Definition 6.12. An abstract nonsingular curve is an open set U of a C_K.

Definition 6.13. A morphism between varieties or abstract nonsingular curves is a continuous map $\phi \colon X \to Y$ such that for any open $V \subseteq Y$ and $f \in \mathcal{O}(V)$ we have $f \circ \phi \in \mathcal{O}(f^{-1}(V))$.

Proposition 6.14. *Every nonsingular quasi-projective curve Y is isomorphic to an abstract nonsingular curve.*

Proof. Just take $K = K(Y)$. For any $p \in Y$, $\mathcal{O}_p$ is a DVR, so we define $Y \xrightarrow{\varphi} C_K$ b $p \mapsto \mathcal{O}_p$. To see it is continuous, we just need to prove the image is open. We may assume Y is affine, since this makes it harder, but writing $Y = k[x_1, \ldots, x_n]/I$, this amounts to prove that there are finitely many R_p that do not contain some of x_i. But we saw this is finite. □

7. 26/09/2019

Proposition 7.1. *X an abstract curve and $p \in X$. Y projective and a morphism $\varphi\colon X - p \to Y$ then it can be extended to p.*

Proof. We can assumne tha $Y = \mathbb{P}^N$. Now choose $U \subseteq X$ such that the image of U does not meet any of the H_i. Now $\frac{x_i}{x_j}$ is regular on the image, and let $f_{ij} = \varphi^*(x_i/x_j)$, which is regular on U. Let $v := v_p(f_{k0})$ be the minimal among all i. Then $v_p(f_{ik})$ is always ≥ 0. Now $\mathbb{A}^n$ with coordinate ring $k[x_0/x_k, \dots, x_N/x_k]$, Then define $q \mapsto (f_{0k}(q), \dots, f_{Nk}(q))$, which is a well-defined morphism of $U \cup \{p\}$ to $\mathbb{P}^N$. □

Theorem 7.2. *If K is a function field of dimension 1 over k, then the absctract curve C_K is isomorphic to a non-singular projective curve.*

Proof. For any $p \in C_K$, we saw there is an affine curve C_p such that $p \in C_p$, and C_p isomorphic to an open set U_p inside C_K. Since C_p is quasi-compact, we can cover C_K by finitely many such $\bigcup U_i = C_K$. Consider the closure Y_i of C_i. Now consider the morphism given by the lemma above $C_K \to Y_i$, and so $C_K \to \prod Y_i$, which is still projective. Take the closure of the image to be Y, and consider $C_K \to Y$.

Y contains the diagonal $\Delta(\bigcap U_i)$, so the function field of Y is the same as that of C_K, that is, $K(Y) = K$. So for any $q \in Y$, take an affine neighborhood with coordinate ring A. Then $K(A) = K$, and so for any maximal ideal of A, there is a DVR R_p which dominates the localization. This means that $p \in C_K$ is mapped to q. So $C_K \to Y$ is surjective.

We want to prove that $\mathcal{O}_p \simeq \mathcal{O}_q$. Let $p \in U_i \subseteq Y_i$. Then if p has image p' in $U_i \subseteq C_K \to \prod Y_i \twoheadrightarrow Y_i$, we have the inclusions $\mathcal{O}_{p',Y_i} \to \mathcal{O}_{q,Y} \to \mathcal{O}_{p,C_K}$, and so these are dominant, so $\mathcal{O}_{p',Y_i} \simeq \mathcal{O}_{p,C_K}$, and so also isomorphic to $\mathcal{O}_{q,Y}$, and this concludes the proof. □

Corollary 7.3. *Every abstract curve is isomorphic to a nonsingular quasi-projective curve. Every curve is birationally equivalent to a non-singular projective curve.*

Corollary 7.4. *The following three categories are equivalent.*

1. *Nonsingular projective curves with dominant morphisms.*
2. *Quasi-projective curves with dominant rational maps.*
3. *Function fields of dimension 1 over k, with k field extensions as morphisms.*

Proof. The only thing that is left is to consider the map (3) $\to$ (1). Consider $K_1 \subseteq K_2$. Then we want to construct a dominant morphism $C_{K_2} \to C_{K_1}$. For any $p_2 \in C_{K_2}$, consider an affine smooth curve U_2 containing it. By (2) $\iff$ (3) we now there is a morphism $U_2 \to U_1$, and we may assume both are affine smooth by shrinking. By the theorem before we can extend this morphism to a morphism $C_{K_2} \to C_{K_1}$. □

7.1. **Schemes.**

Definition 7.5. Given a topological space X, a *presheaf* $\mathcal{F}$ on X is a collection of abelian groups $\mathcal{F}(U)$ for any oepn set U of X, and a colection of morphisms $\rho_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ for any $V \subseteq U$ such that there are compatible and such that $\mathcal{F}(\emptyset) = 0$.

Definition 7.6. $\mathcal{F}$ is a *sheaf* if it is a presheaf and:

(1) If $s \in \mathcal{F}(U)$ and and open cover $\{V_i\}$ such that $s|_{V_i} = 0$, then $s = 0$.
(2) If there is any collection $\{V_i\}$ and elements $s_i \in V_i$ that agree on the intersections, then there is an $s \in \mathcal{F}(\bigcup V_i)$ that restrict to all the s_i.

8. 01/10/2019

Definition 8.1. For a presheaf $\mathcal{F}$, we have the *stalks* $\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U)$.

Proposition 8.2. *If $\mathcal{F} \to \mathcal{G}$ is a morphism between two sheavesm then it is an isomorphism if and only if $\mathcal{F}_p \xrightarrow{\sim} \mathcal{G}_p$.*

Proof. We want to prove $\mathcal{F}(U) \to \mathcal{G}(U)$ is iso. If $s \in \mathcal{F}(U)$ maps to 0, then we know $\varphi(s)_p = 0_p$ for any p. So we can take a cover of U such that $\varphi(s)$ is 0 at all open sets of the cover. Hence $\varphi(s) = 0$. To prove it is surjective, for a $t \in \mathcal{G}(U)$, there is $s_p \in \mathcal{F}_p$ with $t_p = \varphi(s_p)$ for any p. Taking an open cover, we can glue them together to get a $s \in \mathcal{F}(U)$. □

Definition 8.3. For a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of presheaves, we define the presheaves $\ker\varphi, \operatorname{im}\varphi, \operatorname{coker}\varphi$ in the naive way.

Proposition 8.4. *If $\mathcal{F}, \mathcal{G}$ are sheaves, then* $\ker\varphi$ *is a sheaf, but that is not necessarily true for* $\operatorname{coker}\varphi$ *and* $\operatorname{im}\varphi$.

Proposition 8.5. *For any presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ (the sheafification) with a morphism $\theta \colon \mathcal{F} \to \mathcal{F}^+$ such that any morphism to a sheaf factor uniquely through $\mathcal{F}^+$.*

Proof. We define $\mathcal{F}^+(U) = \{s\colon U \to \bigsqcup_{p\in U} \mathcal{F}_p \colon s(p) \in \mathcal{F}_p$ and for any p there is $p \in U,\ t \in \mathcal{F}(U)$ with $s(q) = t_q$ for $q \in U\}$. It is easy to see it is a sheaf, and the morphism $\mathcal{F} \to \mathcal{F}^+$ is the natural one. It is also easy to see that the stalks of $\mathcal{F}$ and $\mathcal{F}^+$ are the same. □

Definition 8.6. We take the image and cokernel sheaves of a morphism by taking the sheafification of the the presheaves.

Definition 8.7. If $f\colon X \to Y$ is a continuous morphism, and $\mathcal{F}$ is a sheaf on X, we define $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. If $\mathcal{G}$ is a sheaf on Y, we define $f^{-1}(\mathcal{G})(U) = \left(\varinjlim_{V\supseteq f(U)} \mathcal{G}(V)\right)^+$.

Definition 8.8. If $i\colon Z \subseteq X$ is a subspace, we write $\mathcal{F}|_Z = i^{-1}\mathcal{F}$, and one can check $(\mathcal{F}|_Z)_p = \mathcal{F}_p$ for $p \in Z$.

8.1. **Schemes.**

Definition 8.9 (Affine Scheme)**.** For a ring A, we define the sheaf $\mathcal{O}$ by

$$\mathcal{O}(U) = \{s\colon U \to \bigsqcup_{p\in U} A_p \colon \text{for any } p \in U \text{ there is } V \subseteq U,\ a, f \in A \text{ such that} \mathfrak{p} \in V \implies f \notin \mathfrak{p} \text{ and } s(q) = a/f\}.$$

Proposition 8.10. *For any $f \in A$, let $D(f) = \operatorname{Spec} A - V(f)$. Then these form a basis for the topology.*

Proof. Let $\mathfrak{p} \in V(\mathfrak{a})$. Choose $f \in \mathfrak{a} - \mathfrak{p}$. Then $D(f) \subseteq \operatorname{Spec} A - V(\mathfrak{a})$. □

Proposition 8.11. *For $\mathfrak{p} \in \operatorname{Spec} A$, we have $\mathcal{O}_p \simeq A_p$. For all $f \in A$, we have $\mathcal{O}(D(f)) = A_f$. In particular $\Gamma(\operatorname{Spec} A) = A$.*

9. 03/10/2019

Proof. We saw that $\mathcal{O}_p \simeq A_p$. To prove the second claim, we have a map $A_f \to \mathcal{O}(D_f)$. To prove it is injective, consider $a/f^n = b/f^m$ in $\mathcal{O}(D_f)$. Let $\mathfrak{a} = \operatorname{Ann}(af^m - bf^n)$. Since for any $\mathfrak{p}$ we have $a/f^n = b/f^m \in A_\mathfrak{p}$, then there is $g \notin \mathfrak{p}$ with $g \in \mathfrak{a}$, that is, $\mathfrak{p} \not\supseteq \mathfrak{a}$. This holds for any $\mathfrak{p} \in D(f)$, so $V(\mathfrak{a}) \subseteq V(f)$. Hence there is k with $f^k \in \mathfrak{a}$, that is, $f^k(af^n - bf^m) = 0$, and hence $a/f^n = b/f^m \in A_f$.

Now let $s \in \mathcal{O}(D_f)$. This means there is an open cover V_i of D_f such that $s|_{V_i} = a_i/h_i$ for $h_i \notin \mathfrak{p}$ for any $\mathfrak{p} \in V_i$. We may assume $V_i = D(h_i)$ for some h_i by what we saw before. So $\bigcup D(h_i) \supseteq D(f)$. This implies $\bigcap V(h_i) \subseteq V(f)$. So $(f) \subseteq \sqrt{\sum h_i}$. So there is n with $f^n = \sum_{i=1}^k g_i h_i$ for $g_i \in A$.

This means that we may assume there are only finitely many h_i. So a_i/h_i are equal in $D(h_ih_j)$, and so we have that they are the same in $A_{h_-h_j}$ by the injection above, so there is p such that $(h_ih_j)^p(h_ia_j - h_ja_i) = 0$, and chosing p to work for all pairs, we have $a_ih_i^p/h_i^{p+1}$ are all equal in the intersections. So now, after replacing the variables, we can assume that the sections are a_i/h_i with $a_ih_j = a_jh_i$. Using that $f^n \in (h_1, \ldots, h_k)$, write $f^n = \sum b_ih_i$, and if $a = \sum b_ia_i$, then one can check $a/f^n = a_i/h_i$. □

Definition 9.1. A *ringed space* is a topological space X with a sheaf of rings.

Definition 9.2. A *morphism* between two ringed spaces $(\varphi, \varphi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is such that φ is continuous and $\varphi^\sharp\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$.

Definition 9.3. A *locally ringed space* is a ringed space such that all stalks are local rings.

Definition 9.4. A morphism between two locally ringed spaces is such that the induced map on the stalks is local, that is, that the pullback of the maximal ideal is the maximal ideal.

Proposition 9.5. *The construction* $A \mapsto (\operatorname{Spec} A, \mathcal{O})$ *is functorial from rings to locally ringed spaces, and it is a full functor.*

Proof. For $\varphi\colon A \to B$, we define $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$, and note $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$. Checking that this is a morphism of locally ringed spaces is not hard. For the fullness, if $(f, f^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$, then it induces $\varphi\colon A = \Gamma(\operatorname{Spec} A, \mathcal{O}_A) \to \Gamma(\operatorname{Spec} A, f_*\mathcal{O}_B) = \Gamma(\operatorname{Spec} B, \mathcal{O}_B) = B$, and we can prove that this is the inverse of the process above, that is, that $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. Since f is a locally ringed morphism, we have the commutativity of

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \longrightarrow & B_{\mathfrak{p}}. \end{array}$$

and this implies that the bottom map is a local morphism. □

Definition 9.6 (Scheme)**.** A *scheme* is a locally ringed space that is covered by affine schemes.

Example 9.7. We can take $X_1 = X_2 = \mathbb{A}^1_k$ and glue them along $\mathbb{A}^1_k - \{0\}$ to get the line with two origins.

9.1. **Analog for projective.** Let $S = \bigoplus_{d\geq 0} S_d$ a graded ring. Consider $S^+ = \bigoplus_{d>0} S_d$, and $\operatorname{Proj} S = \{\mathfrak{p}\colon$ homogeneous primes $\mathfrak{p} \not\supseteq S^+\}$, with topology given as usual.

To define the structure sheaf, we let

$$\mathcal{O}_{\operatorname{Proj} S}(U) = \{s\colon U \to \bigsqcup_{p\in U} S_{(p)} \colon \text{locally given by homogeneous element of degree } 0\}$$

as before.

10. 08/10/2019

Proposition 10.1. *For* $\operatorname{Proj} S$*, we have* $\mathcal{O}_p \simeq S_{(p)}$*, that* $D(f) \simeq \operatorname{Spec} S_{(f)}$ *for* $f \in S^+$*. In particular, it is a scheme.*

Proof. The first part is similar to the affine case. For a homogeneous ideal $\mathfrak{a}$, we define an ideal $\varphi(\mathfrak{a}) = (\mathfrak{a}S_f) \cap S_{(f)}$. This is a map $\varphi\colon D(f) \to \operatorname{Spec} S_{(f)}$. That it is a homeomorphism is easy. To see it is an isomorphism, suffices to check on the stalks, and these are $(S_{(f)})_{(\mathfrak{p})} \simeq S_{(\mathfrak{p})}$. □

Definition 10.2. A morphism $X \to S$ is a scheme X over S.

Proposition 10.3. *For* k *algebraically closed, there is a faithfull functor from varieties over* k *to schemes over* k*.*

Proof. For a variety X, consider the topological space $t(X)$ whose points are he irreducible closed sets of X, and the topology is given by declaring the algebraic sets of X to be closed on $t(X)$. Then we consider the map $P \mapsto \overline{\{P\}}$, and we can check they have the same open sets, and pushing $\mathcal{O}_X$ through this, we can check that $t(x) \simeq \operatorname{Spec} A$. □

10.1. **Basic properties of schemes.**

Definition 10.4. X is connected/irreducible if the topological space is connected/irreducible.

Definition 10.5. X is reduced/integral if for any $U \subseteq X$, $\mathcal{O}_X(U)$ is reduced/integral domain.

Proposition 10.6. X *is integral if and only if* X *is reduced and irreducible.*

Proof. That integral implies reduced is easy. If X is not irreducible, there are two distinct disjoint open sets, and then $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2)$, which is not an integral domain.

Conversely, suppose X is irreducible and reduced. Let $f, g \in \mathcal{O}_X(U)$ with $fg = 0$, we consider $U \subseteq Y \cup Z$ with $Y = \{x\colon f_x \in \mathfrak{m}_x\}$ and $Z = \{x\colon g_x \in \mathfrak{m}_x\}$. These are closed, and since X, hence

U, is irreducible, we must have, say $U \subseteq Y$, which means that $f \in \mathrm{Nil}(U)$, which means it is in the nilradical of all the afine sets of U. Since X is reduced, this means $f = 0$. □

Definition 10.7. X is locally Noetherian if it can be covered by $\operatorname{Spec} A_i$ such that all A_i are Noetherian. X is Noetherian if it is covered by finitely many such A_i. (equivalently, it is locally Noetherian + quasi-compact).

Proposition 10.8. *A scheme if locally Noetherian if and only if all affine opens are Noetherian.*

Proof. Consider a cover $U_i = \operatorname{Spec} B_i$ of X by Noetherian. We can use this to find a topological basis $\operatorname{Spec} A_i$ of Noetherian ones. Let $U = \operatorname{Spec} A$ be any affine open. Then it is covered by open sets as above, so we may assume X is affine. For every A_i, we may find f_i such that $\operatorname{Spec} A_{f_i} \subseteq \operatorname{Spec} A_i$. If $f|_{\operatorname{Spec} A_i} = \overline{f}$, then $A_f = B_{\overline{f}}$ and so A_f is Noetherian.

So now the problem is the following: if $\operatorname{Spec} A$ is covered by $\operatorname{Spec} A_f$ and A_f are Noetherian, then $\operatorname{Spec} A$ is also Noetherian. This means we may consider $(f_1, \dots, f_n) = (1)$ with A_{f_i} Noetherian. Consider $\varphi_i \colon A \to A_{f_i}$. For $\mathfrak{a} \subseteq A$, consider $\bigcap \varphi_i^{-1}\varphi_i(\mathfrak{a})$. It suffices to prove this is $\mathfrak{a}$. Suppose b is an element of such intersection. Then $b = a_i/f_i^{m_i} \in A_{f_i}$ for $a_i \in \mathfrak{a}$. So $f_i^{n_i}(bf_i^{m_i} - a_i) = 0$, so $f_i^{n_i+m_i} b = f_i^{n_i} a_i$, and since $(1) = (f_1^M, \dots, f_n^M)$, we have $b \in \mathfrak{a}$. □

Definition 10.9. A morphism $f \colon X \to Y$ is *locally of finite type* if there exist a covering of Y by $\operatorname{Spec} B_i$ such that $f^{-1} \operatorname{Spec} B_i = \bigcup \operatorname{Spec} A_{ij}$ such that A_{ij} is a finite type B_i-algebra.

Proposition 10.10. *As in the above, this holds for any such cover.*

Definition 10.11. f is of finite type if $j < \infty$.

Example 10.12. If V is a variety over k algebraically closed, then $t(V)$ is integral of finite type over k. This is not enough to define variety because of the line with two origins.

Definition 10.13. $(U, \mathcal{O}_X|U)$ is an open subscheme for every U open.

Definition 10.14. A closed subscheme is $f \colon Y \hookrightarrow X$ closed embedding such that $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

Example 10.15. For $X = \operatorname{Spec} A$, the closed subschemes are $\operatorname{Spec}(A/\mathfrak{a})$. Note the closed subscheme is not defined by the underlying topological space.

11. 10/10/2019

11.1. **Products.** For X, Y schemes over S, we want to consider the product $X \times_S Y$.

Theorem 11.1. *$X \times_S Y$ exists*

Proof. When X, Y are affine, this is just the tensor product.

If $X \times_S Y$ exists and $U \subseteq X$ open, then $U \times_S Y$ exists and is $p_1^{-1}(U)$. This is easy because of the universal property.

Consider an open cover X_i of X. If $X_i \times_S Y$ exists for any i, then we want to prove that $X \times_S Y$ exists. Now $X_{ij} = X_i \cap X_j$ satisfy $X_{ij} \times_S Y = p_{i1}^{-1}(X_{ij})$, and so we can glue them together. In particular, we are done when S is affine.

If S is covered by affines S_i. If X_i, Y_i are the preimages in X, Y, then we can check $X_i \times_{S_i} Y_i \simeq X_i \times_S Y$, and so we are done by a step above. □

Definition 11.2. For a point $y \in Y$, we can consider the residue field $k(y)$, and define the fiber of $X \to Y$ at y to be $X \times_Y \operatorname{Spec} k(y)$.

Proposition 11.3 (Ex 3.15)**.** *Let X be finite type over k. Then the following are equivalent:*

(1) $X \times_k k^s$ is irreducible,

(2) $X \times_k \overline{k}$ is irreducible,

(3) $X \times_k K$ is irreducible for any K.

Then X is called geometrically irreducible over k.

11.2. **Separatedness and properness.** Intuitively, X is separated should be correspondent to Haussdorf, and $X \to Y$ proper should correspond to a proper map of topological spaces (image of compact set is compact).

Example 11.4 (Non-separated)**.** Line with double origin.

Definition 11.5. $X \to Y$ is separated if $X \xrightarrow{\Delta} X \times_Y X$ is a closed immersion.

Proposition 11.6. *If $f : X \to Y$ are two affine schemes, then f is separated.*

Proof. Δ corresponds to $A \otimes_B A \to A$ by multiplication, and this is surjective. □

Corollary 11.7. *$X \to Y$ is separated if and only if $\Delta(X) \subseteq X \times_Y X$ is closed.*

Proof. This is since checking the surjectivity on the map of sheaves is a local question. □

12. 17/10/2019

We will prove the following theorem.

Theorem 12.1 (Valuative criterion for separatedness)**.** *Let X be Noetherian. Then $f\colon X \to Y$ is separated if for any valuation ring R, there is at most one dotted arrow below*

$$\begin{array}{ccc} \operatorname{Spec} K(R) & \longrightarrow & X \\ \big\downarrow & \overset{\cdot^{\cdot^{\cdot}}}{\nearrow} & \big\downarrow \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

Remark 12.2. The intuition is the following. For R a DVR, think of $\operatorname{Spec} R$ as a disk, and $\operatorname{Spec} K(R)$ a punctured disk. Then this means that given a map from a punctured disk to X, there is at most one lift to the entire disk.

Lemma 12.3. *Let R be a valuation ring and X a scheme. For $T = \operatorname{Spec} R$, $U = \operatorname{Spec} K(R)$. Then the data $T \to X$ is the same data as $x_1, x_0 \in X$ with $x_0 \in \overline{\{x_1\}}$, $k(x_1) \subseteq K$ and R dominates $\mathcal{O}_{x_0, \overline{\{x_1\}}}$. (here we see $\overline{\{x_1\}}$ with the reduced structure)*

Proof. Simply unravel the definitions. □

Lemma 12.4. *If $f\colon X \to Y$ is quasi-compact, then the image of X is closed if and only if it is stable under specialization.*

Proof. Consider $y \in \overline{f(X)}$. Take an addine neighborhood $y \in \operatorname{Spec} B \subseteq Y$. Then $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} B \times_Y X = \bigcup_{i=1}^k \operatorname{Spec} A_i$. So $y \in \overline{f(\operatorname{Spec} A_i)}$ for one of the i. For the morphism $B \to A_i$, consider the ideal $\mathfrak{p}' \subseteq B$ corresponding to y. Let $\mathfrak{p}$ be a minimal prime of $A_{\mathfrak{p}'}$. Then $f(x_p) \rightsquigarrow y$. □

Proof of theorem. Assume the map is separated. If h_1, h_2 are two liftings, then there is a map $(h_1, h_2)\colon T \to X \times_Y X$. Then $(h_1, h_2)(U) \subseteq \Delta(X) \subseteq_{\text{closed}} X \times_Y X$. Then $(h_1, h_2)(T) \subseteq \overline{\Delta(X)} = \Delta(X)$. Now we can use the first lemma to prove $h_1 = h_2$.

Now assume we always have unique liftings. Since X is Noetherian, $\Delta(X) \subseteq X \times_Y X$ is quasi-compact. So by the second lemma, we only need to prove $\Delta(X)$ is stable under specialization. Let $y_1 \in \Delta(X)$, and $y_1 \rightsquigarrow y_0$. Consider the reduced subscheme $\overline{\{y_1\}} \subseteq X \times_Y X$, and call $k(y_1) = K$. So $\mathcal{O}_{y_0, \overline{\{y_1\}}} \subseteq K$. So there exist a valuation ring R dominating such ring. Then we have

$$\begin{array}{ccc} T & \longrightarrow & X \times_Y X \\ \big\uparrow & & \big\uparrow \\ U & \longrightarrow & \Delta(X) \end{array}$$

and we get two morphisms $T \to X$ by the two projections, and so by assumption we have that T maps inside $\Delta(X)$, which implies what we want. □

Corollary 12.5. *We assume all schemes are Noetherian.*

(a) Open and closed immersions are separated.
(b) Compositions of separated is separated.
(c) Separatedness is stable under base change.
(d) If $f: X \to Y$ and $f': X' \to Y'$, then $(f, f'): X \times X' \to Y \times Y'$ is separated.
(e) If $f: X \to Y$ and $g: Y \to Z$ such that $g \circ f$ is separated, then f is separated.
(f) $f: X \to Y$ is separated if and only if there is an open cover V_i of Y such that $X \times_Y V_i \to V_i$ is separated for every i. (separated is local on target)

Proof of (c). Let $X' = X \times_Y Y'$. If $h_1, h_2: T \to X'$ and $h: X' \to X$, then we must have that $h \circ h_1 = h \circ h_2$, and by the universal property of X', this implies $h_1 = h_2$. □

Definition 12.6. $f: X \to Y$ is *closed* if and only if for any $Z \subseteq X$ closed, the $f(Z)$ is closed. It is *universally closed* if it is closed after any base change.

Definition 12.7. $X \to Y$ is *proper* if it is separated, of finite type and universally closed.

Example 12.8. $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not proper. Base changing by $\mathbb{A}^1_k$, we have $\mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^1_k$ by $(x, y) \mapsto y$ which is not closed since the image of $Z = V(xy - 1)$ is not closed.

Theorem 12.9 (Valuative criterion for properness)**.** *Let X be Noetherian and $f: X \to Y$ of finite type. Then $f: X \to Y$ is proper if for any valuation ring R, there is exactly one dotted arrow below*

$$\begin{array}{ccc} \operatorname{Spec} K(R) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

Proof. Assume f is proper. We only need to show uniqueness.

$$\begin{array}{ccccc} U & \longrightarrow & X \times_Y T & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ & & T & \longrightarrow & Y \end{array}$$

Let ξ_0, ξ_1 be the point of T. Let x_1 be the image of U in $X \times_Y T$. By properness, the image of $\overline{x_1}$ is T. So let $x_1 \rightsquigarrow x_0$ which maps to ξ_0. Now look at $R \subseteq \mathcal{O}_{x_0, \overline{x_1}} \subseteq k(x_1) = K$. Since R is a valuation ring, then this implies $\mathcal{O}_{x_0,\overline{x_1}} \simeq R$. Now this data gives a morphism $T \to X \otimes_Y T$, which is the lift we wanted.

Now assume the lift always exists and is unique. We need to prove f is universally closed. Let $X' \to Y'$ be a base change, and $Z \subseteq X'$. Since X is Noetherian, f is quasi-compact, and hence if f'. So we only need to prove Z is stable under specialization. So let $y_1 \in f'(Z)$ and $x_1 \in Z$ mapping to it, and $y_1 \rightsquigarrow y_0$, Call $k(x_1) = K$. Then $\mathcal{O}_{y_0,\overline{y_1}} \subseteq k(y_1) \subseteq K$, and choose a valuation ring R dominating it, and apply the criterion to this setting

$$\begin{array}{ccc} U & \longrightarrow & X' \\ \downarrow & \overset{g}{\nearrow} & \downarrow \\ T & \longrightarrow & Y' \end{array}$$

then $y_0 \in \mathrm{Im}(T)$, and so $y_0 \in h(Z)$ since Z is closed. □

13. 22/10/2019

Proposition 13.1. *Assume everything is Noetherian.*

(a) Closed immersions are proper.
(b) Compositions of proper are proper.
(c) Properness is stable under base change.
(d) Products of proper are proper.
(e) Cancelation holds: if $g \circ f$ is proper and g is separated, then f is proper.
(f) Properness is local on the base.

Definition 13.2 (Projective morphism)**.** Let $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$. Then $X \to Y$ is projective if it factors through a closed immersion $X \to \mathbb{P}^n_Y$.

Theorem 13.3. *A projective morphism of Noetherian schemes $X \to Y$ is proper.*

Proof. We may assume $X = \mathbb{P}^n_Y$. Since properness is local on the base, we may assume $Y = \operatorname{Spec} A$.

By induction on n, we may assume $U \to X$ comes from $A[x_1/x_0, \ldots, x_n/x_0] \to K$. Look at $\nu(x_i/x_0) =: s_i$. Then $\nu(x_i/x_j) = s_i - s_j$. Taking the minimal s_j, then $A[x_0/x_j, \ldots, x_n/x_j] \to K$ lies inside R. This is precisely the lifting we want.

To see it is separated is easy. □

Remark 13.4. All smooth projective curves are projective (essentially proven in Chapter 1). Moreover, all smooth proper surfaces are projective, but this is harder, but there is a singular proper surface which is not projective.

Definition 13.5. For k algebraically closed, we call a *variety* a separated integral finite type scheme over k.

Theorem 13.6. *The image of the varieties we studied before are precisely the quasi-projective integral schemes over* k*.*

Theorem 13.7 (Chow's lemma)**.** *Assume* S *Noetherian. Assume* $X \to S$ *is proper. Then there exist a projective* S*-scheme* X' *with a map* $X' \to X$ *that is an isomorphism on a dense Zariski open set of* X*.*

Proof. We may assume that X is irreducible (using Noetherian).

Then for any $x \in X$, we can find an open neighborhood $U_i \subseteq X$ of x such that $U_i = \operatorname{Spec} A_i \to \operatorname{Spec} B_i$ where $\operatorname{Spec} B_i$ is open in S, and A_i is finitely generated over B_i. Now choose a closed immersion $U_i \hookrightarrow \mathbb{A}^n_{B_i}$, and we map this to $\mathbb{P}^n_S$.

Now cover X be finitely many such U_i. Denote X_i the closure of U_i in $\mathbb{P}^n_S$. Now $X_i \to S$ is projective. Let $U = \bigcap U_i$, which is still a dense open set in X. Now look at $U \to X \times X_1 \times \cdots \times X_n$. Now take its closure X'. We have maps $X' \to X$ and $X' \to X_1 \times \cdots X_n$. Let X'' be its image. We want to prove $X' \simeq X''$, as then X' will be projective. This is left as an exercise. □

13.1. **Sheaves.**

Definition 13.8. An $\mathcal{O}_X$-module $\mathcal{F}$ is a sheaf such that $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module with the appropriate compatibilities.

Now if $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$ modules, then the kernel, cokernel, image and quotient are $\mathcal{O}_X$ modules.

Definition 13.9. We have an $\mathcal{O}_X$-module $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ by being the sheatification of the one whose sections are $\operatorname{Hom}(\mathcal{F}(U), \mathcal{G}(U))$.

In the same way, can define $\mathcal{F} \otimes \mathcal{G}$.

Definition 13.10. An $\mathcal{O}_X$ module $\mathcal{F}$ is locally free if there is an open cover such that the restrictions of $\mathcal{F}$ are free.

Definition 13.11. An ideal sheaf is a subsheaf of $\mathcal{O}_X$.

Definition 13.12. For a morphism $X \to Y$ and a $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{O}_Y$-module $\mathcal{G}$, we have an $\mathcal{O}_Y$-module $f_*\mathcal{F}$ (by $\mathcal{O}_Y \to f_*\mathcal{O}_X$).

We also consider $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ an $\mathcal{O}_X$-module.

Moreover, f_*, f^* are an adjoint pair.

Now for $X = \operatorname{Spec} A$ and an A-module M, we define $\tilde{M}$ on X by

$$\tilde{M}(U) = \{s \in \bigsqcup_{p \in U} M_p \colon \text{it is given by an open cover}\}.$$

Proposition 13.13. *$\tilde{M}$ is an $\mathcal{O}_X$-module, and if $p \in X$, $\tilde{M}_p \simeq M_p$. Moreover, $\tilde{M}(D(f)) = M_f$.*

Proposition 13.14. *Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$. Then the functor $M \mapsto \tilde{M}$ is exact and fully faithful, and if M, N are two A-modules, $\widetilde{M \otimes N} = \tilde{M} \otimes \tilde{N}$. Also, $f_*\tilde{N} = \tilde{N}_A$, and $f^*\tilde{M} = (\widetilde{M \otimes_A} B)$*

14. 24/10/2019

Proof. The exactness follows from the exactness at the stalks. To prove it is full, note $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) = \Gamma(X, \mathcal{H}om(\tilde{M}, \tilde{N}))$. Since $\operatorname{Hom}\widetilde{(M, N)} = \mathcal{H}om(\tilde{M}, \tilde{N})$, taking global sections give what we want.

The statement about tensor is trivial by looking at the stalks. □

Definition 14.1. For $(X, \mathcal{O}_X)$ a scheme, a sheaf of $\mathcal{O}_X$-modules is *quasi-coherent* if there is an open cover $U_i = \operatorname{Spec} A_i$ such that $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$. It is *coherent* is M_i are all finitely generated.

Example 14.2. $\mathcal{O}_X$ is coherent, $Y \subseteq X = \operatorname{Spec} A$ given by ideal $\mathfrak{a}$, then $i_*\mathcal{O}_Y = (\widetilde{A/\mathfrak{a}})$.

But for $j\colon U \subseteq X$, $j_!\mathcal{O}_U$ is not quasi-coherent.

For X an integral affine scheme, let $K(U) = \{s \in K\} = \tilde{K}$ is quasi-coherent.

Lemma 14.3. *Let $X = \operatorname{Spec} A$, and $f \in A$ with $D(f) \subseteq X$. If $\mathcal{F}$ is quasi-coherent, then*

(a) if $s \in \Gamma(X, \mathcal{F})$ with $s|_{D(f)} = 0$ then there is n such that $f^n s = 0$.

(b) $t \in \mathcal{F}(D(f))$, then there is n such that $f^n t$ exists in $\mathcal{F}(X)$.

Proof. Suppose X is covered by U_i with $\mathcal{F}|_{U_i} = \tilde{M}_i$. Find $D(g_j) \subseteq U_i$ a basis of the topology. Then restricting $\tilde{M}_i$ to $D(g_j)$, it is also of this form by the previous proposition, we may assume that $U_i = D(g_i)$.

For (a), then $s|_{D(g_i f)} = 0$, which means there is n_i with $f^{n_i} s|_{D(g_i)} = 0$. Since an affine scheme is quasi-compact, we are done.

For (b), there is n_i such that $f^{n_i}s \in \mathcal{F}(D(g_i))$, so again use that affine schemes are quasi-compact. To see that we can glue, let $t_i \in \mathcal{F}(D(g_i))$, and consider $t_{ij} := t_i - t_j$. Then $t_{ij}|_{D(g_i g_j t)} = 0$. By (a), we can then multiply by a power of f to make the gluing work. □

Proposition 14.4. *$\mathcal{F}$ is quasi-coherent if and only if for any* $\operatorname{Spec} A \subseteq X$ *we have* $\mathcal{F}|_{\operatorname{Spec} A} = \tilde{M}$.

If X is Noetherian, then $\mathcal{F}$ is coherent if and only if the same but with M finitely generated.

Proof. Let $U = \operatorname{Spec} A$. Then we can cover U by U_i affine such that $\mathcal{F}|_{U_i} = \tilde{M}_i$. Let $M = \Gamma(U, \mathcal{F})$. This gives a morphism $\tilde{M} \to \mathcal{F}$. In the same way we have morphisms $M_{g_i} \to M_i$. The proposition above means that $M_i = M \otimes A_{g_i}$. Hence $\mathcal{F} \simeq \tilde{M}$.

For the coherent case, we just need to prove that if M_{g_i} are all finitely generated, then M is also finitely generated. □

Corollary 14.5. *For $X = \operatorname{Spec} A$, $M \mapsto \tilde{M}$ gives an equivalence to quasi-coherent sheaves.*

Proposition 14.6. *Let $X = \operatorname{Spec} A$. For an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of $\mathcal{O}_X$-modules and $\mathcal{F}'$ is quasi-coherent, then*

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0.$$

Proof. Let $s \in \Gamma(X, \mathcal{F}'')$. Then for any x, there is $x \in D(f)$ such that $s|_{D(f)}$ can be lifted to t. Then there is some n such that $f^n s$ can be lifted to $\Gamma(X, \mathcal{F})$. This is because the obstructions lie in $\mathcal{F}'$. □

15. 29/10/2019

Proposition 15.1. *Kernel, cokernel, image, extensions of quasi-coherent sheafs are quasi-coherent. If X is Noetherian, the same is true for coherent.*

Proof. The proof is local, so we can assume $X = \operatorname{Spec} A$, and then this becomes a statement about finitely generated modules of A.

For the extension statement, by the previous proposition we have $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$, and so $\Gamma(X, \mathcal{F})$ is finitely generated as an A-module. Then from the morphisms $\widetilde{\Gamma(X, \mathcal{F})} \to \mathcal{F}$, an application of the five-lemma give us that this is an isomorphism. □

Proposition 15.2. *Let $f \colon X \to Y$ a morphism. (a) If $\mathcal{G}$ is quasi-coherent, then $f^*\mathcal{G}$ is quasi-coherent. The same for coherent if X, Y are Noetherian. (b) If either X is Noetherian or f is quasi-compact, separated, and $\mathcal{F}$ is quasi-coherent, then $f_*\mathcal{F}$ is quasi-coherent.*

Proof. We can check (a) locally, and so check this for the case $\operatorname{Spec} A \to \operatorname{Spec} B$, and now it is clear, as $f^*\mathcal{G} = (\widetilde{M \otimes_B A})$ if $\mathcal{G} = \tilde{M}$.

For (b), f_* is only local on the target, so assume $Y = \operatorname{Spec} B$. Cover X by finitely many $U_i = \operatorname{Spec} A_i$. Now $U_i \cap U_j = \bigcup_k U_{ijk}$ for some finite set of k (in the separated case, $U_i \cap U_j$ is affine). Now

$$0 \to f_*\mathcal{F} \to \bigoplus (f_i)_*\mathcal{F}|_{U_i} \to \bigoplus (f_{ijk})_*\mathcal{F}|_{U_{ijk}}.$$

Hence $f_*\mathcal{F}$ is quasi-coherent since it is the kernel of a map between quasi-coherent. □

Definition 15.3. For a closed immersion $Y \hookrightarrow X$, we define the ideal sheaf $\mathcal{I}_Y$ by $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y$. Note $\mathcal{I}_Y$ is quasi-coherent.

Proposition 15.4. *For any quasi-coherent subsheaf $\mathcal{I} \subseteq \mathcal{O}_X$, there is a unique closed subscheme Y with $\mathcal{I} = \mathcal{I}_Y$. If X is Noetherian, then I_Y is coherent.*

Proof. Consider Y to be the support of $\mathcal{O}_X/\mathcal{I}$. Now consider the closed subscheme $(Y, \mathcal{O}_X/\mathcal{I})$. To check uniqueness, we can assume $X = \operatorname{Spec} A$, and this is easy. □

15.1. **Projective setting.** Let S be a graded ring, $X = \operatorname{Proj} S$, and M a graded module.

Example 15.5. $S(n)$ where $S(n)_d = S_{n+d}$.

Definition 15.6. We define $\tilde{M}$ to be

$$\tilde{M}(U) = \{s \colon U \to \bigsqcup_{p \in U} M_{(p)} \mid \text{locally written as } m/f\}.$$

Proposition 15.7. *For any $p \in X$, we have $(\tilde{M})_p = M_{(p)}$, and if $f \in S$, $\tilde{M}|_{D_+(f)} = (\widetilde{M_{(f)}})$ (recall $D_+(f) = \operatorname{Spec} S_{(f)}$). In particular, $\tilde{M}$ is quasi-coherent, and if S is Noetherian and M finitely generated, then $\tilde{M}$ is coherent.*

Definition 15.8. $\mathcal{O}_X(n) = \widetilde{S(n)}$. $\mathcal{O}_X(1)$ is called the *twisting sheaf of Serre.* We write $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Proposition 15.9. *Assume S be generated by S_1 as a S_0-algebra. Then (a) $\mathcal{O}_X(n)$ is invertible, (b) $\widetilde{M(n)} = \tilde{M}(n)$, (c) for T a graded ring generated by T_1 as T_0-algebra and $\varphi \colon S \to T$, then there is $U \subseteq \operatorname{Proj} T$ with a map $U \to \operatorname{Proj} S$ with U corresponding to the ideals $q \in T$ such that $\varphi^{-1}(q) \not\supseteq S_+$ and $f^*\mathcal{O}_X(n) = \mathcal{O}_Y(n)|_U$ and $f_*(\mathcal{O}_Y(n)|_U) = f_*\mathcal{O}_U(n)$.*

Proof. (a) If $f \in S_1$, then $D_+(f) = \operatorname{Spec} S_{(f)}$ and $\mathcal{O}_{D_+(f)} = \widetilde{S(f)}$ and so $\mathcal{O}_{D_+(f)}(n) = \widetilde{\{\text{degree } n \text{ elements of } S_f\}}$. Now the map $a \mapsto f^n a$ identify these two. Now the condition implies that $D_+(f)$ cover X.

(b) follows from $\widetilde{M \otimes N} = \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ and $M(n) = M \otimes S(n)$. □

Definition 15.10. For a $\mathcal{F}$ on $X = \operatorname{Proj} S$, we define

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)).$$

This is a graded S-module by regarding $s \in S_d$ in $s \in \Gamma(X, \mathcal{O}_X(d))$.

16. 31/10/2019

Proposition 16.1. *Let $S = A[x_0, \dots, x_r]$ and $X = \operatorname{Proj} S$. Then $\Gamma_*(\mathcal{O}_X) = S$.*

Proof. For $t \in \Gamma(X, \mathcal{O}(n))$, we have $t_i := t|_{D_+(X_i)} \in S_{x_i}$ is a degree n element, and $t_i = t_j$ agree on the intersections. We need to prove we can glue t_i to an element of S. Looking at them in $S_{x_0 \cdots x_n}$, we have $t \in \bigcap S_{x_i} = S$. □

Lemma 16.2. *Let X be a scheme, $\mathcal{L}$ invertible, and consider for $f \in \Gamma(X, \mathcal{L})$ the set $X_f \subseteq X$ such that $X_f = \{x \colon f_x \notin \mathfrak{m}_x \mathcal{L}\}$. Let $\mathcal{F}$ be a quasi-coherent sheaf. (a) Suppose X is quasi-compact, and $s \in \Gamma(X, \mathcal{F})$ such that $s|_{X_f} = 0$, then there is $n > 0$ such that $f^n \otimes s = 0 \in \Gamma(X, \mathcal{L}^{\otimes n} \otimes \mathcal{F})$. (b) Suppose X can be covered by finitely many affine U_i with $\mathcal{L}|_{U_i}$ such that $U_i \cap U_j$ is quasi-compact. If $s \in \Gamma(X_f, \mathcal{F})$, there is n such that $f^n \otimes s$ can be extended to $\Gamma(X, \mathcal{L}^{\otimes n} \otimes \mathcal{F})$.*

Proof. (a) Since X is quasi-compact, cover X by finitely many $U_i = \operatorname{Spec} A_i$ that trivialize $\mathcal{L}$. Now $f|_{U_i}$ corresponds to a $g_i \in A_i$. Then $0 = s|_{X_f \cap U_i} = s|_{\operatorname{Spec}(A_i)_{g_i}}$. By the affine case there is n_i such that $g_i^{n_i} s = 0$. Choose n to be the maximum of them, and then $f^n \otimes s = 0$.

(b) Is similar to the above, and use (a) to make the intersections agree by increasing n. □

Proposition 16.3. *Let S be a graded ring generated by S_1 and with S_1 a finitely generated S_0-algebra. Let $X = \operatorname{Proj} S$ and $\mathcal{F}$ a quasi-coherent sheaf. Then there is a natural isomorphism*

$$\beta \colon \widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\sim} \mathcal{F}.$$

Proof. First we define β for any $\mathcal{O}_X$-module. Take $f \in S_1$ and look at $D_+(f) = X_f$ (as $f \in \mathcal{O}_X(1)$). For $t \in \Gamma(X_f, \widetilde{\Gamma_*(\mathcal{F})})$, it is of the form m/f^d for $m \in \Gamma(X, \mathcal{F}(d))$. Since $1/f \in \Gamma(D_+(f), \mathcal{O}_X(-1))$, we have $m/f^d \in \Gamma(X_f, \mathcal{F})$. THis means that we can glue these to form the map β.

That it is an isomorphism when $\mathcal{F}$ is quasi-coherent by the previous lemma (injective by (a) and surjective by (b)). □

Corollary 16.4. *Let A be a ring. (a) If Y is a closed subscheme of $\mathbb{P}^r_A$, then there is a homogeneous ideal $I \subseteq S = A[x_0, \dots, x_r]$ such that Y is the closed subscheme determined by I. (b) A scheme Y over $\operatorname{Spec} A$ is projective if and only if it is isomorphic to $\operatorname{Proj} S$ for some graded ring S with $S_0 = A$ and S is S_0-generated by S_1.*

Proof. (a) Y is determined by an ideal sheaf $\mathcal{I}_Y$. Then $\Gamma_*(\mathcal{I}_Y) \subseteq \Gamma_*(\mathcal{O}) = S$, so $\Gamma_*(\mathcal{I}_Y)$ is a graded ideal, which is what we wanted. Now (b) follows easily. □

Definition 16.5. Let $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ be the pullback of $\mathcal{O}(1)$.

Definition 16.6. Consider $X \to Y$. Then $\mathcal{L}$ on X is very ample relative to Y if there is an immersion (open set of closed immersion) $X \to \mathbb{P}^r_Y$ such that $\mathcal{L} = \mathcal{O}(1)|_X$.

Remark 16.7. X is projective over Y if and only if X is proper over Y and there exist a very ample line bundle. (as proper implies that any immersion $X \to \mathbb{P}^r_Y$ is closed)

Definition 16.8. An $\mathcal{O}_X$-module $\mathcal{F}$ is *globally generated* if one can find global sections s_i such that the sheaf at any point is generated by such sections.

Example 16.9. If $\mathcal{F}$ is quasi-coherent on $X = \operatorname{Spec} A$, then it is globally generated. For $X = \operatorname{Proj} S$, and S is generated by S_1 as S_0-algebra, then $O(1)$ is globally generated.

Theorem 16.10 (Serre)**.** *Let X be a projective scheme over a Noetherian A with $\mathcal{O}(1)$ very ample on X/A. If $\mathcal{F}$ is a coherent sheaf, then for all n sufficiently large, $\mathcal{F}(n)$ is globally generated.*

Proof. X can be covered by X_{x_i} by assumption. Then $\mathcal{F}|_{X_{x_i}}$ becomes a coherent sheaf on an affine, so is generated by finitely many elements. As $x_i \in \Gamma(X, \mathcal{O}(1))$, there is n large such that $x_i^n \otimes m$ can be extended to global sections. Then taking m to be these finitely many generators, this proves that there is n large such that $\mathcal{F}(n)$ is globally generated. □

Corollary 16.11. *Any coherent sheaf $\mathcal{F}$ is a quotient sheaf of some $\bigoplus \mathcal{O}(n_i)$ for some $n_i \in \mathbb{Z}$.*

17. 05/11/2019

Theorem 17.1. *Let K be a field, A a finitely generated k-algebra and X a projective shcmee over A. Let $\mathcal{F}$ be a coherent sheaf on X. Then $\Gamma(X, \mathcal{F})$ is a finitely generated A-module.*

Proof. We proved that there is n such that $\mathcal{F}(n)$ is generated by finitely many global sections of $M = \Gamma_*(\mathcal{F})$. Look at the S-submodule $M' \subseteq M$ generated by the sections above. Then $\tilde{M'} \subseteq \tilde{M} = \mathcal{F}$ and $\tilde{M'}(n) = \mathcal{F}(n)$, which implies $\tilde{M'} = \mathcal{F}$. So assume $M = M'$ is finitely generated and $\mathcal{F} = \tilde{M}$.

Now write $M = M^R \supset \cdots \supset M^0 = 0$ with $M^{i+1}/M^i \simeq (S/\mathfrak{p}_i)(n_i)$. From the exact sequence $\tilde{M}^i \to \tilde{M}^{i+1} \to (\widetilde{M^{i+1}/M^i}) \to 0$, take global sections and then we are reduced to prove to the case $M = (S/\mathfrak{p}_i)(n_i)$. We can assume S is integral by replacing S to $S/\mathfrak{p}_i$ and $X = \operatorname{Proj} S$. So it suffices to prove that $\Gamma(X, \mathcal{O}_X(n))$ is finitely generated A-module. Then $S \hookrightarrow \bigcap S_i \hookrightarrow S_{x_0 \cdots x_n}$, and so for any $y \in \Gamma(X, \mathcal{O}_X(n))$, there is m such that $x_i^m y \in S$. So for m sufficiently large we have $S_{\geq m} y \subseteq S_{\geq m}$, and then this is also true by replacing y to a power of y. Choosing $y = x_0^m$, we get $y^i \in (x_0^m)^{-1} S$, and so $S[y]$ is finitely generated, and so y is integral, so $y \in S'$ (the integral closure of S). Since S' is a finite module over S, this means $\Gamma(X, \mathcal{O}_X(n))$ is a finite S_0-module. □

Corollary 17.2. *If $X \to Y$ is a projective morphism between schemes of finite type over k, then if $\mathcal{F}$ is coherent, so is $f_*\mathcal{F}$.*

Proof. Assume Y is affine and apply the theorem above. □

17.1. **Weil divisors.**

Definition 17.3. X is regular in codimension 1 if for any point $p \in X$ of codimension 1, we have $\mathcal{O}_{p,X}$ is regular.

We assume for this section that X is Noetherian integral and regular in codimension 1. $(\star)$

Definition 17.4. A prime divisor on X is the closure of a height 1 prime ideal.

Definition 17.5. The divisor group $\operatorname{Div}(X)$ is the free abelian group generated by prime divisors.

For a prime divisor Y, let η be its generic point. By $(\star)$, $\mathcal{O}_{\eta,X}$ is a DVR. Now for any $f \in K^\times$, we have a well-defined integer $\nu_Y(f)$.

Lemma 17.6. *For any $f \in K^\times$, f has only finitely many zeroes.*

Proof. Choose $\operatorname{Spec} A \subseteq X$ such that f is regular on $\operatorname{Spec} A$. Then $X \setminus \operatorname{Spec} A$ has only finitely many codimension 1 points, since X is Noetherian. Now for $f \in A$ we know that $Z(f)$ contains finitely many codimension 1 components.

Now note that if Y is a zero of f, then Y is either in $Z(f)$ or $X \setminus \operatorname{Spec} A$. □

Definition 17.7. The principal divisors are of the form $(f) := \sum_Y \nu_Y(f)Y$ for some $f \in K^\times$. We say $D \sim D'$ if $D - D'$ is principal.

Definition 17.8. The class group is defined by $\mathrm{Cl}(X) = \mathrm{Div}(X)/\sim$.

Theorem 17.9. *Let A be a Noetherian domain. Then A is a UFD if and only if $\mathrm{Cl}(\operatorname{Spec} A) = 0$ and A is normal.*

Proof. UFD is the same as any height 1 prime being principal, which translates to $\mathrm{Cl}(\operatorname{Spec} A) = 0$.

If $\mathrm{Cl}(\operatorname{Spec} A) = 0$, then there is $f \in K^\times$ giving the prime divisor of $\mathfrak{p}$. Since (f) is effective, we have $f \in \bigcap_{\mathfrak{p}'} A_{\mathfrak{p}'}$ for all $\mathfrak{p}'$ height one, which is simply A because A is normal. Then for $g \in \mathfrak{p}$, we have by the same reasoning that $g/f \in A$, and hence we can conclude (f) is the prime divisor of $\mathfrak{p}$. □

Example 17.10. If A is a Dedekind domain, $\mathrm{Cl}(\operatorname{Spec} A)$ is the ideal class group.

Proposition 17.11. *For $X = \mathbb{P}^n_k$ and D a divisor in $\mathbb{P}^n_k$, then if H is the hypersurface $(x_0 = 0)$, we have $D \sim \deg(D)H$ and that $\deg(f) = 0$ for all $f \in K^\times$. In particular, $\mathrm{Cl}(\mathbb{P}^n_k) \simeq \mathbb{Z}$.*

Proof. This follows easily from the unique factorization in homogeneous elements of $k[x_0, \dots, x_n]$. □

18. 12/11/2019

Proposition 18.1. *Assume X satisfies $(\star)$. Let $Z \subsetneq X$ be a proper closed subset. Let $U = X \setminus Z$. Then (a) $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is surjective, (b) if $\mathrm{codim}_Z X \geq 2$, then $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism, (c) if Z is a prime divisor, then $\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$.*

Proof. (a) is easy: for any prime divisor Y on U, then $\overline{Y}$ is a prime divisor on X that maps to Y. Now note principal divisors get mapped to principal divisors.

(b) Follows from (a) since $\mathrm{Div}(X) = \mathrm{Div}(U)$.

(c) We have $0 \to \mathbb{Z} \to \mathrm{Div}(X) \to \mathrm{Div}(U) \to 0$, and this induces the sequence we want. □

Example 18.2. Let Y be a degree d surve in $\mathbb{P}^2$ and $U = \mathbb{P}^2 \setminus Y$. Then

$$\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^2) = \mathbb{Z} \to \mathrm{Cl}(U) \to 0.$$

Then by what we saw about $\mathbb{P}^n$ we have that $\mathrm{Cl}(U) \simeq \mathbb{Z}/d\mathbb{Z}$.

Example 18.3. Let $A = k[x,y,z]/(xy - z^2)$ and $X = \operatorname{Spec} A$ (cone over a conic). Let Y be given by $y = z = 0$. Now

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus Y) \to 0.$$

Since $X \setminus Y = \operatorname{Spec} A_y$, and $A_y \simeq k[y,z]_y$ is a UFD, so $\mathrm{Cl}(X \setminus Y) = 0$. Hence $\mathrm{Cl}(X)$ is generated by $[Y]$. Also, we have $(y) = 2Y$. Now we see Y is not principal. Since $\mathfrak{m}_0 \subseteq A$ has $\dim(\mathfrak{m}_0/\mathfrak{m}_0^2) = 3$, and y, z span a 2-dimensional subspace, the ideal (y, z) cannot be principal in A.

Proposition 18.4. *Let X satisfying ($\star$). Then $X \times \mathbb{A}^1$ also does, and* $\mathrm{Cl}(X) \simeq \mathrm{Cl}(X \times \mathbb{A}^1)$.

Proof. If $Y \subseteq X \times \mathbb{A}^1$ is such that the image to X is a divisor Z, then $Y = Z \times \mathbb{A}^1$. We know the localization at $\eta(Z)$ is a DVR R, and then the localization at $\eta(Y)$ will be $R[t]_{\eta(Y)}$, which is a DVR since $R[t]$ is regular. Now let $Y \subseteq X \times \mathbb{A}^1$ such that the image of Y in X is the entire X. This case is easier. Hence ($\star$) holds for $X \times \mathbb{A}^1$.

We have a morphism $\mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{A}^1)$. The image is the divisor of the first type (vertical divisors). Now note that for a horizontal divisor D, we have $D|_{\operatorname{Spec} K[t]} = (f)$ since $K[t]$ is a UFD, and then $D - (f)$ is a vertical divisor. Hence the map is surjective.

For injection, we just evaluate the function at $t = 0$ (in fact the functions are already in K, since they are units). $\square$

Example 18.5. Consider $Q = (xy = zw) \subseteq \mathbb{P}^3$. We have that $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in the Segre embedding. Then $\mathrm{Cl}(\mathbb{P}^1) \to \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathrm{Cl}(\mathbb{A}^1 \times \mathbb{P}^1)$ is an isomorphism, and the kernel of the second map is generated by $* \times \mathbb{P}^1$, which is the image of the pullback of the first projection. Hence $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Example 18.6. Let X be a smooth cubic surface in $\mathbb{P}^3$. Then $\mathrm{Cl}(X) \simeq \mathbb{Z}^7$.

Example 18.7. Consider $Q \subseteq \mathbb{P}^3$. We want to define a morphism $\mathrm{Cl}(\mathbb{P}^3) \to \mathrm{Cl}(Q)$. For a prime divisor D with $Q \not\subseteq D$, then we cover $\mathbb{P}^3$ by affine spaces such that $D|_{\mathbb{A}^3_i} = (f_i)$, and then we can associate a divisor $(f_i)|_{Q \cap \mathbb{A}^3_i}$. Divisors can be translated to not contain Q. Taking the generator to be $(x = 0)$, we have the two lines $(x = z = 0)$ and $(x = w = 0)$, and so the image will be $(1, 1)$ under the Segre embedding.

Example 18.8. Consider $C = (t^3, u^3, t^2u, tu^2)$. Then $C \subseteq Q$, and $Q \cap (yz = w^2) = C \cup (y = w = 0)$, and so C has class $(1, 2)$ in $\mathrm{Cl}(Q)$. So C cannot be given as the intersection of Q with a surface.

19. 14/11/2019

Definition 19.1. For $k = \overline{k}$, a curve over k is an integrl separated finite type scheme over k of dimension 1. We say X is *complete* is X is proper, and *smooth* if all local rings are regular.

Proposition 19.2. *Let X be a nonsingular curve over k. Then the following are equivalent: (1) X is projective, (2) X is complete, (3) $X = t(C_K)$.*

Proof. We have already seen (3) $\implies$ (1) $\implies$ (2). The remaining implication follows at once from the valuative criterion of properness. □

Proposition 19.3. *Let X be a complete nonsingular curve and Y any curve over k, and $f\colon X \to Y$ a morphism. Then either $f(X) = p \in Y$ or $f(X) = Y$. In the second case, $K(X)$ is a finite extension of $K(Y)$ and Y is complete.*

Proof. We now $f(X) \subseteq Y$ is closed since X is proper and Y is separated. Since the image is irreducible, it follows that it is either a point or surjective. If $f(X) = Y$, then $K(Y) \hookrightarrow K(X)$ is a field extension. Since they have the same transcendental degree, it is finite. Y is also complete since the image of proper is proper. □

Definition 19.4. For $f\colon X \to Y$ a finite morphism of curves, we define $\deg(f) = [K(X)\colon K(Y)]$.

For X smooth and D a divisor on X, we have $D = \sum n_i P_i$ and we call $\deg(D) = \sum n_i$.

Definition 19.5. Let $f\colon X \to Y$ be a finite morphism between smooth curves. Then we define $f^*\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ the pullback on the level of divisors. It is given by

$$f^*(\sum n_P P) = \sum_{Q \in X} m(Q|f(Q)) \cdot n_{f(Q)} Q$$

where m is the valuation of a parameter of $\mathcal{O}_{f(Q)}$ on $\mathcal{O}_Q$

Proposition 19.6. *Let $f\colon X \to Y$ be a finite morphism of nonsingular curves. Then for any D on Y, we have* $\deg(f^*D) = \deg(f)\deg(D)$.

Proof. We need to show that for $P \in Y$, that $\sum_{f(Q)=P} m(Q|P) = \deg(f)$. But looking locally around P, we can consider an affine map $A \to B$ such that $\mathcal{O}_{p,Y} \simeq A_p$. As A_p is a DVR and B_p is a torsion-free module, then B_p is a free module, and its rank is the degree of f. Now we note that $\dim_{A_p/p}(B_p \otimes A_p/p) = \sum m$, by the Chinese remainder theorem. □

Proposition 19.7. *Let X be a smooth complete curve. Then* $\deg(f) = 0$ *for any* $f \in K(X)$.

Proof. Any $f \in K(X)$ gives a morphism $X \xrightarrow{\pi} \mathbb{P}^1$, and then $(f) = \pi^*(0) - \pi^*(\infty)$, and this proves that $\deg(f) = 0$.

Completeness of X is necessary for π to be finite. □

Hence for a complete smooth curve, we have a well-defined map $\mathrm{Cl}(X) \to \mathbb{Z}$ given by the degree. We call its kernel $\mathrm{Cl}^0(X)$.

Corollary 19.8. *X is a rational curve if and only if there are $P \neq Q$ with $P \sim Q$.*

Example 19.9. Consider the nonsingular curve $E\colon y^2z = x^3 - xz^2$ in $\mathbb{P}^2$. We will prove $\mathrm{Cl}^0(E) \simeq E(k)$.

Proof. Choose the point $P_0 = (0, 1, 0)$, and define the map $E(k) \to \mathrm{Div}^0(E)$ by $P \mapsto P - P_0$. By the previous corollary, this is injective since E is not $\mathbb{P}^1$. The tangent line through P_0 intersects P_0 three times, and so $3P_0 \sim P + Q + R$ for three points P, Q, R in a line. This can be used to reduce any divisor to the form $P - P_0$. □

19.1. **Cartier divisors.** Let X be a Noetherian separated scheme. Consider $\operatorname{Spec} A \subseteq X$, and S the set of nonzero-divisors of A. Call $K = S^{-1}A$, the *total ring* of A.

Thinking of K as a sheaf, we can consider the presheaf $U \mapsto S^{-1}(U)\Gamma(U, \mathcal{O}(U))$. Sheafifying, we call it $\mathcal{K}$, and we have sheafs of abelian groups $\mathcal{K}^*$ and $\mathcal{O}^*$.

Definition 19.10. For a Noetherian separated scheme X, a *Cartier divisor* is an element of $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$.

Proposition 19.11. *Let X be an integral separated scheme, and assume ($\star$). Then there is a morphism* $\mathrm{Cartier}(X) \hookrightarrow \mathrm{Div}(X)$, *and is isomorphic if all local rings are UFDs. This induces a map on the class groups.*

20. 19/11/2019

Proof. Given a Cartier divisor, we can find a finite open cover U_i such the Cartier divisor is given by f_i in U_i. Define D_i to be (f_i). Then $D_i - D_j = (f_i/f_j) = 0$. This means that they glue to a divisor D on X.

Now given a Weil divisor D, for any $p \in X$ we have $D|_{\operatorname{Spec} \mathcal{O}_{p,X}} = (f)$ is principal (as $\mathcal{O}_{p,X}$ is a UFD). So there is a neighborhood U of p such that f is defined on U. Then $D - (f) \subseteq U$ has no components in p. Shrinking U further, we have $D|_U = (f)|_U$. Taking a cover like so, we have a Cartier divisor. □

20.1. **Invertible sheaves.**

Definition 20.1. An invertible sheaf is a locally free sheaf of rank 1.

Definition 20.2. The Picard group $\operatorname{Pic}(X)$ is the group of invertible sheaves up to isomorphism.

Note that we have $\operatorname{Pic}(X) \simeq \mathrm{H}^1(X, \mathcal{O}^\times)$, and so the Cartier class group injects to $\operatorname{Pic}(X)$.

Proposition 20.3. *Let X be integral Noetherian separated. Then $\operatorname{Pic}(X)$ is the same as the Cartier class group.*

Proof. In the cohomology language, $\mathcal{K}^\times$ is flasque in the case X is integral Noetherian seaprated, to $\mathrm{H}^1(X, \mathcal{K}^\times) = 0$.

The image of the Cartier class group is the invertible sheafs that are subsheaves of $\mathcal{K}^\times$. If $\mathcal{L}$ is an invertible sheaf, we have $\mathcal{L} \otimes \mathcal{K} \simeq \mathcal{K}$ locally, and this is an isomorphism globally since X is integral. Hence $\mathcal{L} \subseteq \mathcal{K}$. □

Corollary 20.4. *If $X \simeq \mathbb{P}^n_K$, then all invertible sheafs are isomorphic to some $\mathcal{O}_X(n)$.*

Definition 20.5. A Cartier divisor (U_i, f_i) is *effective* if $f_i \in \Gamma(U, \mathcal{O}_{U_i})$.

Proposition 20.6. *If $D \in \operatorname{Cartier}(X)$ is effective, then it corresponds to a subscheme, and then $\mathcal{I}_D \simeq \mathcal{L}(-D)$.*

20.2. **Projective morphisms.** Given a morphism $\varphi\colon X \to \mathbb{P}^n_A$, we can consider $x_i \in \Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$ for $i = 0, \dots, n$, and pullback to $\mathcal{L} = \varphi^*(\mathcal{O}(1))$. Then $s_i = \varphi^*(x_i) \in \Gamma(X, \mathcal{L})$ generate $\mathcal{L}$.

Proposition 20.7. *Let A be a ring, X a scheme over A. If $\mathcal{L}$ is an invertible sheaf with generators $s_0, \dots, s_n$, then there exist a morphism $\varphi\colon X \to \mathbb{P}^n_A$ with $\varphi^*(\mathcal{O}(1)) = \mathcal{L}$ and $\varphi^*(x_i) = s_i$.*

Proof. Let $X_i = \{p\colon (s_i)_p \notin \mathfrak{m}_p\mathcal{L}\}$. This an open set, and we define a morphism $X_i \to U_i = \operatorname{Spec} A[x_0/x_i, \dots, x_n/x_i]$ by the map on global sections $x_j/x_i \mapsto s_j/s_i$. Now it is clear that they glue together. □

Example 20.8. For any $\varphi \in \operatorname{Aut}(\mathbb{P}^n_k)$, we have $\varphi^*(\mathcal{O}(1)) = \mathcal{O}(1)$ since it generated the Picard group and has a section. So φ is determined by a choice of generators of $\mathcal{O}(1)$, and these are in correspondence with $\mathrm{PGL}(n+1)$.

Example 20.9. In general, for an invertible sheaf $\mathcal{L}$ and set of sections s_i, we can define a morphism from $X \setminus Z$ where $Z = \{p\colon s_i \in \mathfrak{m}_p\mathcal{L}\}$.

Proposition 20.10. *If $\varphi\colon X \to \mathbb{P}^n_A$ is given by sections s_i, then φ is a closed immersion iff X_i are affine and $A[x_0/x_i, \dots, x_n/x_i] \to \Gamma(X_i, \mathcal{O}_{X_i})$ are surjective.*

21. 21/11/2019

Proposition 21.1. *Let $k = \overline{k}$ and X a projective scheme over k. Consider $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ that induce a $\varphi\colon X \to \mathbb{P}^n_k$. Let $V \subseteq \Gamma(X, \mathcal{L})$ be spanned by s_i. Then φ is a closed immersion iff (1) V separates points, in the sense that for $p \neq q$, there is $s \in V$ such that $s \notin \mathfrak{m}_p\mathcal{L}$ and $s \in \mathfrak{m}_q\mathcal{L}$ and (2) V separates the tangent directions, in the sense that for $p \in X$, $\{s \in V\colon s \in \mathfrak{m}_p\mathcal{L}\}$ spans $\mathfrak{m}_p\mathcal{L}/\mathfrak{m}_p^2\mathcal{L}$.*

Proof. (1) is equivalent to φ being injective.

Since X is projective, $X \to \varphi(X)$ is closed, and so (given (1)) is a homeomorphism. We have to show $\mathcal{O}_{\varphi(x),\mathbb{P}^n} \to \mathcal{O}_{x,X}$ is a surjection. The residue fields are isomorphic and the dual of tangent spaces also map isomorphically. Together with being a finitely generated morphism this implies it is surjective: By Nakayama the maximal ideal maps exactly to the maximal ideal, and by Nakayama again this implies it is surjective. □

21.1. **Ample invertible sheaves.**

Definition 21.2. An invertible sheaf $\mathcal{L}$ is *ample* on a Noetherian scheme X if for any coherent sheaf $\mathcal{F}$ we have $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for all $n >> 0$.

Example 21.3. If X is affine, then any coherent sheaf is globally generated.

Example 21.4. Very ample are ample in projective spaces.

Proposition 21.5. *Let $\mathcal{L}$ be an invertible sheaf on X. Then the following are equivalent: (1) $\mathcal{L}$ is ample, (2) $\mathcal{L}^{\otimes m}$ is ample for any $m > 0$, (3) there is $m > 0$ such that $\mathcal{L}^{\otimes m}$ is ample.*

Proof. The only problem is (3) $\implies$ (1). Let $\mathcal{F}$ be a coherent sheaf. Look at $\mathcal{F} \otimes \mathcal{L}^{\otimes i}$ for $i = 0, \dots, m-1$ and we are done. □

Theorem 21.6. *Let X be a finite type scheme over a Noetherian ring A. Let $\mathcal{L}$ be an invertible sheaf. The $\mathcal{L}$ is ample if and only if there exist a sufficiently large m such that $\mathcal{L}^{\otimes m}$ is very ample over A for some m.*

Proof. The converse follows from the previous proposition once we prove very ample implies ample. Let $X \hookrightarrow \mathbb{P}^n_A$ an immersion. Then X may not be closed (if it was, Serre's lemma). Consider $X \hookrightarrow \overline{X} \hookrightarrow \mathbb{P}^n_A$. Then by Ex 5.15 we can extend a coherent sheaf $\mathcal{F}$ of X to one in $\overline{X}$. Then Serre's lemma on $\overline{X}$ gives us what we wanted.

Let $p \in X$ and U an affine neighborhood trivializing $\mathcal{L}$. Let $Z = X \setminus U$. Consider n_Z such that $\mathcal{I}_Z \otimes \mathcal{L}^{\otimes n_Z}$ is globally generated. So find a section s with $s \notin \mathfrak{m}_p \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n_Z}$. Consider the open set X_s. Then $X_s \subseteq U$. As U is affine, we have $X_s \simeq U_f$ where $f = s|_U$, hence X_s is also affine. Since X is Noetherian, we can cover X by finitely many such X_s, and choose $n = n_Z$ for all of them. Let $X_{s_i} = \operatorname{Spec} A[b_{i1}, \ldots, b_{ij}]$. We know that for each b_{ij} there is n such that $b_{ij} s_i^n$ extends to a section of $\Gamma(X, \mathcal{L}^n)$. Choosing n large again, we can take all such sections and use them to embed into a projective space. □

Example 21.7. Consider $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$. Then $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$ and if (a, b) is an element with $a, b > 0$, we have $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{a+1} \times \mathbb{P}^{b+1} \to \mathbb{P}^N$ is a closed embedding. These are precisely the very ample ones and ample ones.

Example 21.8. Consider $y^2 = x^3 - x$ in projective space. Then $3P_0$ is very ample, but P_0 is only ample.

21.2. **Linear systems.** If $\mathcal{L} \subseteq \mathcal{K}$, for a section s we can define the zero locus $(s)_0$.

22. 26/11/2019

Let X be a nonsingular (will be used only to identify Weil and Cartier divisor, most things still work with Cartier divisors) projective variety over $k = \overline{k}$.

Proposition 22.1. *Let D_0 be a divisor on X and $\mathcal{L}$ the associated invertible sheaf. For $s \in \Gamma(X, \mathcal{L})$ nonzero, we have $(s)_0 \sim D_0$, and every effective divisor equivalent to D_0 comes from such a s.*

Definition 22.2. A complete linear system on a nonsingular projective variety is defined as the set of all effective divisors linear equivalent to a given divisor D_0, denoted by $|D_0|$. Note $|D_0| \simeq \mathbb{P}(\Gamma(X, \mathcal{L}))$.

Definition 22.3. A linear system is a subspace $V \subseteq \Gamma(X, \mathcal{L})$, with corresponding $|V| \subseteq |D_0|$.

Definition 22.4. $p \in X$ is a base point of a linear system δ if $p \in D$ for all $D \in \delta$.

Lemma 22.5. *Let δ be a linear system on X corresponding to $V \subseteq \Gamma(X, \mathcal{L})$. Then $p \in X$ is a base point if and only if $s_p \in \mathfrak{m}_p\mathcal{L}$ for all $s \in \delta$. In particular, δ is base point free if and only if $\mathcal{L}$ is generated by elements of δ.*

Remark 22.6. δ separate points if and only if for any $P \neq Q$ there is $D \in \delta$ with $P \in D$, $Q \notin D$. δ separates tangent directions if and only if for any p and $t \in (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$ there is $p \in D \in \delta$ and $t \notin \operatorname{Im}((\mathfrak{m}_{p,D}/\mathfrak{m}_{p,D}^2)^\vee \to (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee)$.

Definition 22.7. For $Y \subseteq X$ and a δ on X, we consider the restricted linear system $\delta|_Y$, which correspond to the image under $\Gamma(X, \mathcal{L}) \to \Gamma(Y, f^*\mathcal{L})$.

22.1. **Proj sheaves and blow-up.** We assume for this section that X is a Noetherian scheme and a quasi-coherent $\mathscr{I} = \bigoplus_{d \geq 0} \mathscr{I}_d$ an $\mathcal{O}_X$-algebra. Assume that $\mathscr{I}_0 = \mathcal{O}_X$ and $\mathscr{I}_1$ is a coherent sheaf and generates $\mathscr{I}$ as a $\mathcal{O}_X$-algebra.

Definition 22.8. We define $\pi\colon \operatorname{Proj} \mathscr{I} \to X$ such that for $U = \operatorname{Spec} A \subseteq X$, we have $\pi|_U\colon \operatorname{Proj} S \to \operatorname{Spec} A$ where $\widetilde{S_d} = \mathscr{I}_d|_U$. Moreover, the invertible sheaves $\mathcal{O}(1)$ glue to $\operatorname{Proj} \mathscr{I}$.

Lemma 22.9. *Let $\mathcal{L}$ be an invertible sheaf on X. Let $\mathscr{I}' = \mathscr{I} \otimes \mathcal{L}$ defined by $\mathscr{I}'_d = \mathscr{I}_d \otimes \mathcal{L}^d$. Then $i\colon \operatorname{Proj} \mathscr{I} \xrightarrow{\sim} \operatorname{Proj} \mathscr{I}'$ but $\mathcal{O}(1)' = i^*\mathcal{O}(1) \otimes \pi'^*\mathcal{L}$.*

Proposition 22.10. *$\operatorname{Proj} \mathscr{I} \to X$ is a proper morphism. If X is quasi-projective with ample $\mathcal{L}$, then $\operatorname{Proj} \mathscr{I} \to X$ is projective and $\mathcal{O}(1) \otimes \pi^*(\mathcal{L}^{\otimes n})$ is very ample for $n \gg 0$.*

Proof. The first part is easy since we can check locally. Since $\mathcal{L}$ is ample, $\mathscr{I}_1 \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n \gg 0$. □

Remark 22.11. Using the definition of projective in EGA, $\operatorname{Proj} \mathscr{I} \to X$ is always projective. So $\mathcal{O}^{\oplus m} \twoheadrightarrow \mathscr{I} \otimes \mathcal{L}^{\otimes n}$, and taking proj we get $\operatorname{Proj} \mathscr{I} \simeq \operatorname{Proj}(\mathscr{I} * \mathcal{L}^{\otimes n}) \subseteq \mathbb{P}^m_X$.

Definition 22.12. Let X Noetherian and $\mathcal{E}$ a locally free sheaf of rank $r+1$. We define $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ to be $\operatorname{Proj} \bigoplus_{d \geq 0} \operatorname{Sym}^d \mathcal{E}$.

Proposition 22.13. *Let $X, \mathbb{P}(\mathcal{E})$ be as above. Then $\pi_*\mathcal{O}(l) \simeq \operatorname{Sym}^l \mathcal{E}$, and also $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{O}(1)$.*

Let X be Noetherian an $\mathcal{F}$ an ideal sheaf, and consider $\mathscr{I} = \bigoplus_{d \geq 0} \mathcal{F}^d$.

Definition 22.14. We define the blow up along $\mathcal{F}$ to be $\operatorname{Bl}_Z X := \operatorname{Proj} \mathscr{I}$.

Definition 22.15. Given $f\colon Y \to X$ and $\mathcal{F}$ is an ideal sheaf on X, we consider $f^{-1}\mathcal{F} \subseteq \mathcal{O}_Y$ to be the image of $f^*\mathcal{F}$ in $\mathcal{O}_Y$.

Proposition 22.16. *If $\pi\colon \mathrm{Bl}_{\mathcal{F}}X \to X$, then $\pi^{-1}\mathcal{F}$ is invertible. Moreover, π is isomorphic outside $Z(\mathcal{F})$.*

Proof. This is because $\pi^{-1}\mathcal{F} \simeq \mathcal{O}(1)$. □

23. 03/12/2019

Proposition 23.1 (Universal property of blow up)**.** *Let $\mathcal{F}$ be an ideal sheaf on a Noetherian X. If $f\colon Z \to X$ is such that $f^{-1}\mathcal{F}\cdot\mathcal{O}_Z$ is invertible, then there is a unique $\tilde{f}$ such that*

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{f}} & \mathrm{Bl}_{\mathcal{F}}X \\ & \searrow^{f} & \downarrow \\ & & X \end{array}$$

Proof. We may assume $X = \operatorname{Spec} A$, so that $\mathcal{F} = \tilde{I}$ for some $I \subseteq A$. Choose generators $a_0, \dots, a_n$ of I, giving a surjection $A[x_0, \dots, x_n] \to \bigoplus_{i \geq 0} I^i$, so that we have $\mathrm{Bl}_{\mathcal{F}}X \subseteq \mathbb{P}^n_A$.

Consider the invertible sheaf $\mathscr{L} := f^{-1}\mathcal{F}\cdot\mathcal{O}_Z$. Define s_i as the image of a_i. The pullback of x_i to Z generate $\mathscr{L}$, so we have $\varphi\colon Z \to \mathbb{P}^n_A$ by $x_i \mapsto s_i$, with $\varphi^*\mathcal{O}(1) = \mathscr{L}$. Now it is easy to check that this factors through the blow up.

To prove uniqueness, we have $\mathscr{L} = \tilde{f}^{-1}(\pi^{-1}\mathcal{F}\cdot\mathcal{O}_{\mathrm{Bl}_{\mathcal{F}}X})\cdot\mathcal{O}_Z = \tilde{f}^{-1}\mathcal{O}(1)\cdot\mathcal{O}_Z$. Since $\tilde{f}^*\mathcal{O}(1)$ surjects into that, we must have $g^*\mathcal{O}(1) \simeq \mathscr{L}$, and we can see this is given by the construction above. □

Corollary 23.2. *Let $Y \to X$ and $\mathcal{F}$ an ideal sheaf on X, and $\mathcal{F}_Y$ the corresponding ideal sheaf in Y. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Bl}_{\mathcal{F}_Y}Y & \longrightarrow & \mathrm{Bl}_{\mathcal{F}}X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Moreover, if $Y \to X$ is a closed embedding, so is the upper arrow.

Proposition 23.3. *If X is a variety over k an $\pi\colon \tilde{X} \to X$ is a blow up, then $\tilde{X}$ is a variety, π is proper, surjective and birational. If X is quasi-projective, then $\tilde{X}$ is quasi-projective.*

Proof. Integrality is clear from the construction. Since π is proper, surjective and finite type, this implies $\tilde{X}$ is a variety. It is birational since it is isomorphic outside $V(\mathcal{F})$. If X is quasi-projective, then π is projective, and so $\tilde{X}$ is quasi-projective. □

Theorem 23.4. *Let X a quasi-projective variety, and Z another variety with* f*: $Z \to X$ projective birational. Then f is a blow up.*

Proof. Since Z is projective, consider $g\colon Z \hookrightarrow \mathbb{P}^n_X$ and let $\mathscr{L} := g^*\mathcal{O}(1)$. Now we find e such that $\bigoplus_{d\geq 0} f_*\mathscr{L}^{\otimes de}$ is generated by degree 1 elements. Since X is Noetherian, this is a local question. When $X = \operatorname{Spec} A$, let $S = A[x_0, \dots, x_n]$ and $Z = \operatorname{Proj} S/I$. Let $T = \bigoplus_{d\geq 0} f_*\mathscr{L}^{\otimes d}$. The map $S \to T$ is an isomorphism in high enough degree. This implies there is such e. (This is the same as changing $\mathbb{P}^n_X$ via the e-tuple embedding, and we assume this in what follow)

Now $Z = \operatorname{Proj} \bigoplus_{d\geq 0} f_*\mathscr{L}^{\otimes d}$. Since Z is integral, $\mathscr{L} \subseteq \mathscr{K}_Z$. Then $f_*\mathscr{L} \subseteq f_*\mathscr{K}_Z = \mathscr{K}_X$ since f is birational. Consider an ideal sheaf $\mathcal{F} \subseteq \mathcal{O}_X$ the denominator of $f_*\mathscr{L}$

$$\mathcal{F}(U) = \{a \in \mathcal{O}(U) \colon a \cdot f_*\mathscr{L}(U) \subseteq \mathcal{O}(U)\}.$$

Since X is quasi-projective, it has an ample line bundle $\mathscr{M}$. Choose e such that $\mathscr{M}^{\otimes e} \otimes \mathcal{F}$ has a nonzero section. Then $\mathscr{M}^{-e} \to \mathcal{F}$, and this is an injection since X is integral. Then $\mathscr{M}^{-e} \otimes f_*\mathscr{L}$ maps to an ideal sheaf $\mathscr{I}$. Then $\operatorname{Proj} \bigoplus_{d\geq 0} \mathscr{I}^d = Z$ since $\mathscr{M}^{-e}$ is invertible. □

23.1. **Differentials.**

Definition 23.5. For B an A-algebra, we say a A-module morphism $d\colon B \to M$ for a B-module M is a *derivation* if it is linear, $d(bb') = bd(b') + d(b)b'$ and $d(a) = 0$.

Definition 23.6. The *module of relative differentials* $B \to \Omega_{B/A}$ satisfies the universal property that any derivation $B \to M$ factors uniquely through $\Omega_{B/A}$.

Proposition 23.7. *Consider $f\colon B \otimes_A B \to B$ and let $I = \ker f$. Then $d\colon B \to I/I^2$ given by $b \mapsto 1 \otimes b - b \otimes 1$ is isomorphic to $\Omega_{B/A}$.*

Proposition 23.8. *The formation of $\Omega_{B/A}$ commutes with tensor products and localizations.*

Proposition 23.9 (First exact sequence)**.** *For an exact $A \to B \to C$, we have an exact sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

Proposition 23.10 (Second exact sequence)**.** *Given $A \to B$ and $C = B/I$, we have an exact sequence as C-modules*

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

where the first map is given by $I \otimes B/I \to \Omega_{B/A} \otimes B/I$ maping $b \in I$ to $d(b) \otimes 1$.

Theorem 23.11. *If K is a finitely generated field over k, then $\dim_K \Omega_{K/k} \geq \text{tr.deg}(K/k)$. Equality holds if and only if K/k is separable.*

24. 05/12/2019

Proposition 24.1. *Let B be a local rings containing its residue field k. Then $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$ is an isomorphism.*

Proof. By the second exact sequence, we have $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k \to 0$. We show that the dual is surjective. Indeed, the dual map is $\text{Hom}_B(\Omega_{B/k}, k) \to \text{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$, and the first is $\text{Der}(B, k)$. Now given $h \colon \mathfrak{m}/\mathfrak{m}^2 \to k$, consider the derivation δ that takes $b = c + a$ with $a \in k, c \in \mathfrak{m}$ to $\delta(b) = h(c)$. □

Theorem 24.2. *Let B be a local ring containing its residue field k. Assume $K = K(B)$ is separable over k and that B is the localization of a finitely generated k-algebra. Then $\Omega_{B/k}$ is a free B-module of rank$=$ $\dim B$ if and only if B is a regular local ring.*

Proof. $\implies$ is trivial by the last proposition.

For the converse, by the last proposition we have $\dim_k \Omega_{B/k} \otimes_B k = \dim B$. Let $K = K(B)$. Then $\Omega_{B/k} \otimes_B K = \Omega_{K/k} \geq \text{tr.deg}(K/k)$ with equality if and only if K/k is separated. From the k part, by Nakayama we have $0 \to N \to B^{\dim B} \to \Omega_{B/k} \to 0$, and tensoring with K gives that N is torsion, and hence is trivial. □

Definition 24.3. For $f \colon X \to Y$, consider the diagonal $X \xrightarrow{\Delta} X \times_Y X$, and $U \subseteq X \times_Y X$ an open set with $X \hookrightarrow U$ a closed immersion. If $\mathcal{F}$ is the ideal sheaf of this immersion, $\mathcal{F}/\mathcal{F}^2$ does not depend on U, and we define $\Omega_{X/Y} = \Delta^*(\mathcal{F}/\mathcal{F}^2)$. This is the *sheaf of differentials*.

Remark 24.4. Affine locally, this gives the Kähler differentials. Moreover, $\Omega_{X/Y}$ is quasi-coherent. If Y is Noetherian and f is finite type, then $\Omega_{X/Y}$ is coherent.

Remark 24.5. From the discussion above, if $X' = X \times_{Y'} Y$, and $g \colon X' \to X$, then $g^*\Omega_{X/Y} = \Omega_{X'/Y'}$.

Proposition 24.6. *For $f\colon X \to Y$ and $g\colon Y \to Z$, we have*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

and if $W \hookrightarrow X$ closed with ideal $\mathscr{I}$, then

$$\mathscr{I}/\mathscr{I}^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

Example 24.7. Let $X = \mathbb{A}^n_A$ and $Y = \operatorname{Spec} A$. Then $\Omega_{X/Y}$ is a free $\mathcal{O}_X$-module generated by dx_i.

Proposition 24.8. *Let $X = \mathbb{P}^n_A$ and $Y = \operatorname{Spec} A$. Then*

$$0 \to \Omega_{X/Y} \to \mathcal{O}(-1)^{n+1} \to \mathcal{O} \to 0.$$

Proof. Let $S = A[x_0, \ldots, x_n]$. Let $E = S(-1)^{n+1}$, with generators $e_0, \ldots, e_n$ with degree 1. Considering the map $e_i \to x_i$, we get $0 \to M \to E \to S \to 0$. Let U_i be the standard open sets. Then M_{x_i} is a free module generated by $\frac{1}{x_i}e_j - \frac{x_j}{x_i^2}e_i$. Now the differentials at U_i are $d(x_j/x_i) = \frac{dx_j}{x_i} - \frac{x_j}{x_i^2}dx_i$ formally, and so the formal map $e_i \to dx_i$ show that $\frac{1}{x_i}e_j - \frac{x_j}{x_i^2}e_i \mapsto d(x_j/x_i)$ glues to a global map. □

Definition 24.9. X is *nonsingular* if it is locally Noetherian and all local rings are regular.

Theorem 24.10. *Let R be a local regular ring. Then $R_\mathfrak{p}$ is still local regular for any prime $\mathfrak{p}$.*

Corollary 24.11. *X is nonsingular if and only if it is regular in all closed points.*

Theorem 24.12. *Let X be finite type, irreducible and separated over an algebraically closed field. Then $\Omega_{X/k}$ is locally free of rank equals to $\dim X$ if and only if X is nonsingular over k.*

Proof. Let $x \in X$ be a closed point, and $B = \mathcal{O}_{x,X}$. Then $\Omega_{X/k} \otimes_{\mathcal{O}_X} B = \Omega_{B/k}$, and the statement follows from the previous corollary and the local study we did before. □

Corollary 24.13. *If X is a variety over algebraically closed field k, then there is a nontrivial open U where it is nonsingular.*

Proof. $\Omega_{X/k} \otimes K = \Omega_{K/k}$ is free of dimension $\dim X$. Since $\Omega_{X/k}$ is coherent, then there is $U \subseteq X$ such that $\Omega_{U/k}$ is locally free of rank $\dim X$. □

Theorem 24.14. *Let X be nonsingular and $Y \subseteq X$ an irreducible closed subscheme. Let $\mathcal{F}$ be the ideal sheaf. Then Y is nonsingular if and only if (1) $\Omega_{Y/k}$ is locally free and (2) $0 \to \mathcal{F}/\mathcal{F}^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$.*

Moreover, in the above case then $\mathcal{F}$ is locally generated by $r := \operatorname{codim}(Y, X)$ elements, and $\mathcal{F}/\mathcal{F}^2$ is a locally free sheaf of rank r on Y.

Proof. Assume (1) and (2). Let $n = \dim X$. By (2) and Nakayama, on a closed point $\mathcal{F}$ is generated by $n - q$ elements where $q = \operatorname{rank}(\Omega_{Y/k})$. So $\dim Y \geq n - (n - q) = q$. Now $\Omega_{Y/k} \otimes k \simeq \mathfrak{m}_Y/\mathfrak{m}_Y^2$, and so $\dim Y \leq q$. Hence Y is nonsingular. Moreover, we proved the second statement.

For the converse, the kernel of $\varphi\colon \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k}$ is locally free $\mathcal{O}_Y$-module of rank $r := \dim X - \dim Y$. So we can choose $x_1, \dots, x_r \in \mathcal{F}$ such that $dx_1, \dots, dx_r$ generate $\ker\varphi$. Choose Y' to be the vanishing locus of $x_1, \dots, x_r$. Then $Y \subseteq Y'$. Consider the sequence $\mathcal{F}'/\mathcal{F}'^2 \to \Omega_{X/k} \otimes \mathcal{O}_{Y'} \to \Omega_{Y'/k}$ where $\mathcal{F}'_x = (x_1, \dots, x_r)$. Then this is actually exact. Then Y' satisfies (1) and (2) and hence is nonsingular. And $\dim Y = \dim Y'$. So $Y = Y'$. □

Theorem 24.15 (Bertini)**.** *If X is nonsingular and $X \subseteq \mathbb{P}^n$ closed, then there exist a hyperplane H such that $X \cap H$ is regular at every point.*

25. 10/12/2019

Proof. For $x \in H$, consider $B_x = \{H \ni x \colon X \subseteq H \text{ or } X \cap H \text{ not regular at } x\}$. Fix f_0 giving H_0 such that $x \notin H_0$. For any H corresponding to f, consider $\varphi_x \colon H \mapsto f/f_0 \in \mathcal{O}_{x,X}$. Now B_x correspond to the preimage of $\mathfrak{m}_x^2$ under φ_x. Let V be the dual $\mathbb{P}^n$. Consider $B \subseteq X \times V$ given by $\bigcup\{x\} \times B_x$. We want to prove $B \to V$ is not surjective. We will do this by showing that $\dim B < \dim V$. This is since B is a $\mathbb{P}^{n-r-1}$ bundle over X where $r = \dim X$ (since X is nonsingular). So $\dim B = n - 1$. □

Definition 25.1. We let $\omega_X = \bigwedge^d \Omega_X$.

Theorem 25.2. *If X and X' are nonsingular projective and birational, then $\Gamma(X, \omega_X^{\otimes n}) = \Gamma(X', \omega_{X'}^{\otimes n})$ for all $n \geq 0$.*

Proof. Choose $V \subseteq X$ the maximal open set such that the birational morphism φ is defined on V. Then $\varphi^*\omega_{X'} \to \omega_V$ is an isomorphism on a open set. This defines a map $\Gamma(X', \omega_{X'}^{\otimes n}) \to \Gamma(V, \omega_V^{\otimes n})$, and is injective since a section cannot be zero in an open set. Now note the analogous map for X

is bijective. Indeed, $\operatorname{codim}_{X\setminus V} X \geq 2$ since X is proper. Doing the same changing X and X' gives what we want. $\square$