18.705: COMMUTATIVE ALGEBRA, FALL 2019

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Contents

Problem sets

Altman-Kleiman

Website

1. 04/09/2019

Goal: study *interesting* comutative rings using the language of category theory, and geometryic intuition when appropriate.

Rings are commutative.

Example 1.1. The following are rings

(1) $\{0\}$,
(2) fields,
(3) rings of integers in number fields,
(4) $k[t]$ for a (algebraically closed) field k (regular functions on the affine line $\mathbb{A}^1$), $k[t_1, \dots, t_n]$, quotients, etc.

1.1. Universal properties.

Definition 1.2. Given a ring A and an ideal $I \subseteq A$, the *quotient ring* is a ring A/I, equipped with ring homomorphism $A \to A/I$, sending I to 0, and such that for any other wing B and a map $A \to B$ killing I, then there is a unique map $f \colon A/I \to B$ such that

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ & \searrow & \downarrow f \\ & & B \end{array}$$

Remark 1.3. By general category theory such A/I is unique up to unique isomorphism.

Definition 1.4. Let A be a ring. The polynomial algebra in one variable over A is an A-algebra $A[t]$ equipped with a distinct element $t \in A[t]$ such that for any other such pair (B, b), there is a unique map $f\colon A[t] \to B$ with

$$\begin{array}{ccccc} A & \longrightarrow & A[t] & & t \\ & \searrow & \downarrow & & \downarrow \\ & & B & & b \end{array}$$

Remark 1.5. It generalizes to any set indexing the variables.

Definition 1.6. Let A_1, and A_1 be rings. The *product* ring $A_1 \times A_2$ equipped with homomorphisms $p_i\colon A_1 \times A_2 \to A_i$ for $i \in \{1, 2\}$ such that for an other such ring P, there is a unique map $f\colon P \to A_1 \times A_2$ such that

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{p_1} & \downarrow f & {}^{p_2}\searrow & \\ & & A_1 \times A_2 & & \\ & \swarrow^{p_1} & & {}^{p_2}\searrow & \\ A_1 & & & & A_2 \end{array}$$

Remark 1.7. Generalizes to any index set I.

Remark 1.8. In $A_1 \times A_2$, one has idempotents $e_1 := (1, 0)$ and $e_2 := (0, 1)$. Conversely, if A is a ring with an *idempotent* $e \in A$, then

$$\begin{aligned} A &\xrightarrow{\sim} A_1 \times A_2 \\ a &\longmapsto (ae, a(1 - e)) \end{aligned}$$

where $A_1 := Ae$ and $A_2 := A(1 - e)$ with identities e and $1 - e$.

Warning 1.9. There is no such thing as a *direct sum* of rings ($A \to A \times B$ is not a ring homomorphism).

2. 6/09/2019

Definition 2.1. Given ideals I and J, we define the *colon ideal* or *transporter*

$$(I\colon J) = \{a \in A\colon aJ \subseteq I\}.$$

Definition 2.2. A subset $S \subseteq A$ is *multiplicative* if S is closed under finite products.

Definition 2.3. An ideal $\mathfrak{p}$ is *prme* if $A \setminus \mathfrak{p}$ is multiplicative.

Definition 2.4. We define the *spectrum* $\operatorname{Spec} A$ to be the set of all prime ideals.

Proposition 2.5. *A ring homomorphism $A \to B$ induces a map $\operatorname{Spec} B \to \operatorname{Spec} A$.*

Proposition 2.6. *An ideal $I \subset A$ is maximal if and only if A/I is a field, and prime if and only if A/I is a domain.*

Definition 2.7. Given a domain A, its *fraction field* $\operatorname{Frac}(A)$ is universal among fields F equipped with an injective ring homomorphsim $A \to F$.

Definition 2.8. Given a domain A and $P = A[t_i]$, we denote the *order of vanishing* $\operatorname{ord} f$ to the minimum of the degree of the monomials.

Definition 2.9. If K is a field, we denote the *field of rational functions* $K(t_i)$ to be the fraction field of $K[t_i]$.

3. 09/09/2019

Lemma 3.1. *Let S be a multiplicative subset of A and I an ideal with $I \cap S = \emptyset$. Let $\mathfrak{J} = \{J \supseteq I \colon J \cap S = \emptyset\}$. Then $\mathfrak{J}$ has a maximal element $\mathfrak{p}$, and any such $\mathfrak{p}$ is a prime ideal.*

Proof. For any such $\mathfrak{p}$, and any $x \notin \mathfrak{p}$, we have $(\mathfrak{p}+(x)) \cap S \neq \emptyset$. So $A \setminus \mathfrak{p} = \{x \in A \colon p+(x) \cap S \neq \emptyset\}$. Now the claim is that $A \setminus \mathfrak{p}$ is a multiplicative set. □

3.1. Radicals.

Definition 3.2. Let $I \subseteq A$ be an ideal. Then $\sqrt{I} := \{x \in A \colon x^n \in I \text{ for some } n\}$.

Example 3.3. Is $\mathfrak{p}$ is prime, then $\sqrt{\mathfrak{p}} = \mathfrak{p}$.

Theorem 3.4. *We have*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I \text{ prime}} \mathfrak{p}.$$

Proof. If $\mathfrak{p} \supseteq I$, then $\sqrt{I} \supseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$. For the other inclusion, consider $x \notin \sqrt{I}$. Apply the lemma with $S = \{1, x, x^2, \ldots\}$, which is disjoint to I. Then we get a prime ideal $\mathfrak{p} \supseteq I$ disjoint to S. This means $x \notin \mathfrak{p}$. □

Definition 3.5. For the ideal 0, $\operatorname{nil} A := \sqrt{0}$ is called the *nilradical* of A, which is the intersection of all prime ideals.

Definition 3.6. A ring A is *reduced* if $\sqrt{0} = 0$.

Definition 3.7. The *Jacobson radical* of a ring is $\operatorname{rad} A := \bigcap_{\mathfrak{p} \text{ maximal}} \mathfrak{p}$.

Remark 3.8. Later we will prove that if A is a finitely generated algebra over a field or over $\mathbb{Z}$, then $\operatorname{nil} A = \operatorname{rad} A$, which is a version of the Nullstellensatz.

Proposition 3.9. *If $u \in A^\times$, then $u + \operatorname{rad} A \subseteq A^\times$.*

Proof. If $u + v \notin A^\times$, then it belong to a maximal ideal. □

Corollary 3.10. *We have* $\operatorname{rad} A = \{x \in A \colon 1 - ax \in A^\times$ *for all* $a \in A\}$.

Proof. $\subseteq$ is clear. Now if $x \notin \operatorname{rad} A$ it is not in some maximal ideal $\mathfrak{m}$. Then $\mathfrak{m} + (x) = A$, so we have $m + ax = 1$ for $m \in \mathfrak{m}$, which means $1 - ax = m$ is not a unit. □

Example 3.11. Consider $A = k[[t]]$ for a field k. Consider $\mathfrak{m} = tk[[t]]$. Note $A^\times = A \setminus \mathfrak{m}$, and that this means A is local, and all ideals are $\mathfrak{m}^n$ for varying $n \geq 0$. So $\operatorname{rad} A = \mathfrak{m}$ and $\operatorname{nil} A = 0$.

Definition 3.12. A ring A with exactly one maximal ideal is called a *local ring*, and the quotient $A/\mathfrak{m}$ is called its *residue field*.

Definition 3.13. A ring A with finitely many maximal ideals is called a *semilocal ring*.

3.2. **Spectrum.**

Definition 3.14. We turn $\operatorname{Spec} A$ into a topological space by taking the closed sets to be

$$V(I) := \{\mathfrak{p} \text{ prime} \colon \mathfrak{p} \supseteq I\}$$

for ideals $I \subseteq A$. Note $V(I) \cup V(J) = V(I \cap J)$ and $\bigcap_i V(I_i) = V(\sum_i I_i)$. This is called the *Zariski topology*.

Proposition 3.15. $V(I) = \emptyset \iff I = A$ *and* $V(I) = \operatorname{Spec} A \iff A \subseteq \operatorname{nil} A$. *Moreover,* $V(I) = V(J) \iff \sqrt{I} = \sqrt{J}$.

Example 3.16. Let k be an algebraically closed field and $A = k[t]$. Then $\operatorname{Spec} A$ is $k \cup \{\eta\}$ with the profinite topology on k and whose all nonempty open sets contain η, that is $\overline{\{\eta\}} = \operatorname{Spec} A$.

4. 11/09/2019

Definition 4.1. For any subset $S \subseteq \operatorname{Spec} A$, we say a point $p \in \operatorname{Spec} A$ is a *generic point for* S if $S = \overline{\{p\}}$.

Proposition 4.2. *A point* $\mathfrak{p} \in \operatorname{Spec} A$ *is closed if and only if* $\mathfrak{p}$ *is a maximal ideal.*

Proof. This is since $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. □

Example 4.3. Let $A = k[x, y]$ for k algebraically closed. Then the points of $\operatorname{Spec} A$ correspond to the ideals of the form $(x - a, y - b)$, to irreducible polynomials and to the zero ideal.

Definition 4.4. A topological space is irreducible if it is nonempty and is not a union of two proper closed subsets.

Proposition 4.5. *The irreducible closed subsets of* $\operatorname{Spec} A$ *are precisely the sets of the form* $V(\mathfrak{p})$ *for primes* $\mathfrak{p}$.

Proof. It is easy to see that $V(\mathfrak{p})$ is irreducible. Conversely, consider $V(I)$ with I radical and not prime. If $ab \in I$ but $a, b \notin I$, then $V(I) = V(I + (a)) \cap V(I + (b))$. □

Definition 4.6. An *irreducible component* of a topological space X is a maximal irreducible (closed) subset. The analogous notion in commutative algebra is of a *minimal prime*.

Theorem 4.7. *Let* I *be an ideal of* A. *Then*

$$\sqrt{I} = \bigcap_{\mathfrak{p}} \mathfrak{p}$$

where $\mathfrak{p}$ *varies over minimal primes over* I.

Proof. Obvious from the previous version and Zorn's lemma. □

5. 13/09/2019

Proposition 5.1. A *is a domain if and only if* A *is reduced and* $\operatorname{Spec} A$ *is irreducible.*

Proof. If A is a domain it is clear. Otherwise, Since $\operatorname{Spec} A$ has only on component, it has only one minimal prime, and it must be $\sqrt{0} = 0$. Hence 0 is prime, that is, A is a domain. □

Proposition 5.2. Spec *is a contravariant functor.*

5.1. **Modules.**

Definition 5.3. The *kernel* of f is a pair (K, i) universal among the pairs such that $K \xrightarrow{i} M \xrightarrow{f} N$ is 0. The *cokernel* is (C, q) universal among the ones such that $M \xrightarrow{f} N \xrightarrow{q} C$ is 0. The *image* of f is the kernel of the cokernel, and the *coimage* is the cokernel of the kernel, and they are isomorphic.

Theorem 5.4. *If A is a PID, then any submodule of a free A-module is free.*

Proof. Let F be a submodule of the free module E. By the axiom of choice, we may take a well ordered I with $E = \bigoplus_{i \in I} A$, with projections π_i. Let $E_i = \bigoplus_{j \leq i} A$, and $F_i = F \cap E_i$. Then $\pi_i(F_i) = (c_i)$ for some $c_i \in A$. If $c_i \neq 0$, we choose $f_i \in F_i$ such that $\pi_i(f_i) = c_i$. Then it is easy to see that f_i form a basis for F. □

6. 16/09/2019

Proposition 6.1. *Given an exact sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} P \to 0$, the following are equivalent:*

(1) *it is isomorphic to $0 \to N \to N \oplus P \to P \to 0$,*
(2) *there exist a section $s\colon P \to M$ such that $\beta \circ s = \mathrm{id}$,*
(3) *there exist a retraction $r\colon M \to N$ such that $r \circ \alpha = \mathrm{id}$.*

Lemma 6.2 (Snake lemma)**.** *Given a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma''} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

with exact rows, we get an exact sequence

$$0 \to \ker(\gamma') \to \ker(\gamma) \to \ker(\gamma'') \to \operatorname{coker}(\gamma') \to \operatorname{coker}(\gamma) \to \operatorname{coker}(\gamma'') \to 0.$$

Definition 6.3. A *presentation* of M is an exact sequence $F_1 \to F_0 \to M \to 0$ with F_0, F_1 free. Then M is *finitely generated* if F_0 can be chosen with finite rank, and *finitely presented* is F_0 and F_1 can be chosen of finite rank.

Example 6.4. $A = k[t, u]$ and $M = (t, u)$ then $A \to A^2 \to M \to 0$ with $z \mapsto (uz, -tz)$ and $(x, y) \to xt + yu$.

7. 18/09/2019

Definition 7.1. For a ring A and a module M, we have functors

$$\operatorname{Hom}(M, \cdot)\colon \operatorname{Mod}_A \to \operatorname{Mod}_A \quad \text{and} \quad \operatorname{Hom}(\cdot, M)\colon \operatorname{Mod}_A \to \operatorname{Mod}_A^{\mathrm{op}}.$$

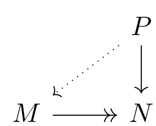
Definition 7.2. For categories $\mathscr{C}, \mathscr{D}$, an *additive functor* $F\colon \mathscr{C} \to \mathscr{D}$ is such that the maps $\operatorname{Hom}(A, B) \to \operatorname{Hom}(FA, FB)$ are group homomorphisms.

Definition 7.3. For contravariant functors $F\colon \mathscr{C} \to \mathscr{D}$, we define left/right exact by applying the definition to $F\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

Theorem 7.4. *Both Hom functors are left exact.*

Theorem 7.5. *Let A be a ring, P a module. The following are equivalent, and when they are true we say P is* projective.

(1) *Given $M \twoheadrightarrow N$ and $P \to N$, there is a $P \to M$ that is compatible.*



(2) *Every short exact sequence $0 \to K \to M \to P \to 0$ splits.*

(3) *P is a direct summand of a free module.*

(4) *$\operatorname{Hom}(\cdot, P)$ is exact.*

Proof. (1) $\implies$ (2): Use 1 with $P = N$ to get a section.

(2) $\implies$ (3): Take M free and use (2).

(3) $\implies$ (4): Suppose $K \oplus P$ is free. Then $\operatorname{Hom}(K \oplus P, \cdot)$ is exact, and $\operatorname{Hom}(K \oplus P, \cdot) = \operatorname{Hom}(K, \cdot) \oplus \operatorname{Hom}(P, \cdot)$, so both must be exact, which implies (4).

(4) $\implies$ (1): Apply $\operatorname{Hom}(P, \cdot)$ to $0 \to K \to M \to N \to 0$, and so in particular $\operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N) \to 0$, which is what we want. $\square$

8. 23/09/2019

Categories, colimits.

9. 25/09/2019

More on colimits.

10. 27/09/2019

Proposition 10.1. *For a colimit* $\varinjlim_i M_i$ *in modules of* A *and* N *is an* A*-module, apply* $\mathrm{Hom}(N, \cdot)$ *to the limit. Then we have a map* $\theta\colon \varinjlim_i \mathrm{Hom}(N, M_i) \to \mathrm{Hom}(N, \varinjlim_i M_i)$.

Now suppose the limit is filtered.

(1) If N *is finitely generated, then* θ *is injective.*

(2) If N *is finitely presented, then* θ *is an isomorphism.*

Proof of (1). Let $n_1, \dots, n_m$ be generators of N. Let $f \in \varinjlim_i \mathrm{Hom}(N, M_i)$. Say it is represented by $f\colon N \to M_i$. If $\theta(f) = 0$, then $\theta(f)(n_k) = 0$, and this holds in $\varinjlim_i M_i$, so there is a j_k with $f(n_k) = 0$ in M_{j_k}, so there is a $j > j_k$ such that $f = 0 \in M_j$, so $f = 0 \in \varinjlim_i M_i$. □

Theorem 10.2. *Filtered colimits preserve exactness.*

Proof. The morphisms $\varinjlim_i L_i \to \varinjlim_i M_i \to \varinjlim_i N_i$ come from the universal properties and their composition is 0. If $m \in \varinjlim_i M_i$ is in the kernel, then it is $m \in M_i$, and since it maps to 0, we can further choose i such that $M_i \to N_i$ maps m to 0. This means m is in the image of $L_i \to M_i$, and so that it is in the iamge of $\varinjlim_i L_i \to \varinjlim_i M_i$. □

11. 30/09/2019

Tensors.

Theorem 11.1. *For* N *a* (A, B)*-bimodule,* $\otimes_A N$ *and* $\mathrm{Hom}_B(N, \cdot)$ *is an adjoint pair.*

Corollary 11.2. $\otimes_A N$ *preserves colimits, and is right exact.*

Theorem 11.3 (Watts 1960)**.** *Any* A*-linear functor* $F\colon \mathrm{Mod}_A \to \mathrm{Mod}_A$ *that preserves direct sums and cokernels is isomorphic to* $\otimes_A P$ *for some* $P = F(A)$*. In general, if* F *does not preserves direct sums and cokernels, then we only have a natural transformation* $\Theta\colon \otimes_A P \to F$ *such that* $\Theta(A)$ *is the identity.*

12. 02/10/2019

Definition 12.1. For a ring A and an A-module M, let TM, $\bigwedge M = TM/(m \otimes m)$ and $\mathrm{Sym} M = TM/(a \otimes b - b \otimes a)$ be the tensor algebra, the exterior algebra, and the symmetric algebra.

Example 12.2 (Determinant)**.** If M is a free A-module of rank r, then $\det M := \bigwedge^r M$ is free of rank 1. If $f\colon M \to M$ is a linear map, this induces a map $\bigwedge^r f\colon \bigwedge^r M \to \bigwedge^r M$, and this is multiplication by the determinant.

12.1. **Flatness.**

Definition 12.3 (Serre, $\sim$1955)**.** The functor $\otimes_A M$ is always right exact. We call M *flat* if it is also left exact.

Lemma 12.4. *$\bigoplus_{i \in I} M_i$ is flat if and only if all M_i are flat.*

Proof. Follows from the distributivity of $\oplus$ and $\otimes$. □

Corollary 12.5. *Projective modules are flat.*

Remark 12.6. $\otimes_A M$ is faithful if and only if $N \otimes M \to P \otimes M$ being 0 implies $N \to P$ to be 0.

Definition 12.7. M is *faithfully flat* if M is flat and faithful.

Example 12.8. Nonzero free modules.

Proposition 12.9. *Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ with M_3 flat. Then tensoring with any N preserves the exactness. Moreover, M_1 is flat if and only if M_2 is flat.*

13. 04/10/2019

Proof. Take a presentation $0 \to K \to F \to N$ and tensor the initial sequence, use snake lemma on the first two rows.

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & M_1 \otimes K & \longrightarrow & M_2 \otimes K & \longrightarrow & M_3 \otimes K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 \otimes F & \longrightarrow & M_2 \otimes F & \longrightarrow & M_3 \otimes F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & M_1 \otimes N & \longrightarrow & M_2 \otimes N & \longrightarrow & M_3 \otimes N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

For the second part, again take any injection $N' \to N$ and tensor it with the initial sequence and apply the snake lemma. □

13.1. **Geometric interpretation of flatness.** For a prime $\mathfrak{p} \subseteq A$, let $k(\mathfrak{p}) = \mathrm{Frac}(A/\mathfrak{p})$. We have $A \twoheadrightarrow A/\mathfrak{p} \hookrightarrow k(\mathfrak{p})$, which induces $\operatorname{Spec} k(\mathfrak{p}) \to \operatorname{Spec} A/\mathfrak{p} \to \operatorname{Spec} A$. Now $\operatorname{Spec} A/\mathfrak{p}$ is an irreducible closed subset of $\operatorname{Spec} A$, with $\operatorname{Spec} k(\mathfrak{p})$ mapping to its generic point.

Now given a module M, we a family of vector spaces $M \otimes_A k(\mathfrak{p})$. M being flat is kinda of equivalent to $M \otimes k(\mathfrak{p})$ being well-behaved over $\mathfrak{p}$. Faithfully flat is kinda of equivalent to $M \otimes k(\mathfrak{p})$ being well-behaved and non zero (if M is a A-algebra, this is the same as the projection being surjective).

Theorem 13.1 (Lazard's Theorem). *M is flat if and only if it is a filtered colimit of free modules of finite rank.*

13.2. **Cayley–Hamilton.**

Theorem 13.2 (Cayley–Hamilton). *Let A be a ring, and $X \in M_n(A)$. Then* $\det(TI - X)(X) = 0$

Proof. Prove for the generic matrix by choosing an embedding to $\mathbb{C}$ and diagonalizing. □

Theorem 13.3 (Determinant trick). *For A a ring and a finitely generated module M with an endomorphism $\phi\colon M \to M$, Choose a basis of M and write ϕ as a matrix X. Then $P_X(\phi) = 0 \in$* $\mathrm{End}(M)$.

Proof. Just note that it is true for free modules, and hence for its quotients. □

14. 07/10/2019

Lemma 14.1. *Let M be a finitely generated A-module, and $I \subseteq A$ an ideal. If $M = IM$, then there is $b \in I$ such that $(1+b)M = 0$.*

Proof. Choose generators m_i of M, and write relations $m_i = \sum_j a_{ij} m_j$ with $a_{ij} \in I$. Then use Cayley–Hamilton. □

Lemma 14.2 (Nakayama's lemma)**.** *Let A be a ring, $I \subseteq \operatorname{rad}(A)$ be an ideal. Then for a finitely generated module M, $M = IM$ implies $M = 0$.*

Proof. Apply the lemma. There is $b \in I$ with $(1+b)M = 0$. But $1+b$ is a unit, hence $M = 0$. □

Corollary 14.3. *Let A be a local ring with maximal ideal $\mathfrak{m}$, and M a finitely generated module. Consider $m_1, \ldots, m_n$. Then m_i generate M if and only if they generate $M/\mathfrak{m}M$.*

Proof. Let N be the module generated by the m_i. Let $Q = M/N$. Then we have an exact sequence $N/\mathfrak{m}N \to M/\mathfrak{m}M \to Q/\mathfrak{m}Q \to 0$. Now by Nakayama, $Q/\mathfrak{m}Q = 0$ if and only if $Q = 0$, which is what we want. □

Remark 14.4. There is an analogue in group theory: if G is a p-group, g_i generate G if and only if they generate the Frattini quotient $G/(G^p[G,G])$.

Proposition 14.5. *Let A be a ring and M an A-module. Then free $\implies$ projective $\implies$ flat. Moreover, they are all equivalent if A is local and M is finitely presented.*

Also, for a projective module, finitely generated $\implies$ finitely presented.

Proof. Supose A is local and $\mathfrak{m}$ is the maximal ideal, with $k = A/\mathfrak{m}$. Let M be a flat finitely presented A-module. Choose a basis $\overline{m_i}$ for $M \otimes k$, and choose lifts m_i. By Nakayama, m_i generate M. Since M is finitely generated, there are finitely many m_i. So we have a map $A^n \to M \to 0$. Since M is finitely presented, we may choose this such that $0 \to L \to A^n \to M \to 0$ with L finitely generated. Tensoring with k, and using that M is flat, we conclude $L \otimes k = 0$. By Nakayama again, we conclude $L = 0$.

For the second claim, let M be finitely generated projective. Doing the same thing as above, $0 \to L \to A^n \to M \to 0$ is split. Then $A^n \to L$ giving the splitting is surjective, and so L is finitely generated. □

14.1. **Integral extensions.**

Definition 14.6. A a ring, B an A-algebra. Then $x \in B$ is integral if x satisfies a monic equation.

Proposition 14.7. *x is integral if and only if $A[x]$ is generated as a A-module, if and only if x is contained in an A-subalgebra $C \subseteq B$ that is finitely generated as a A-module.*

15. 09/10/2019

Proof. All is easy, except that the last one implies integrality, but this follows from the determinant trick. □

Definition 15.1. The integral closure (or normalization) of A over B is the ring of integral elements over B.

Example 15.2. $A = k[t^2, t^3]$ is not normal.

16. 11/10/2019

17. 16/10/2019

18. 18/10/2019

We will establish

$$M \text{ has property } P \iff M_{\mathfrak{m}} \text{ has property } P \text{ for all maximal ideals } \mathfrak{m}$$

for several P.

Proposition 18.1. *$P =$"is trivial".*

Proof. Take $s \in M$ nonzero, and consider a maximal ideal that contains $\mathrm{Ann}(s)$. □

Proposition 18.2. *$P =$"exactness".*

Proof. Follows from the above and the exactness of localization. □

Proposition 18.3. *$P =$"flatness".*

Proof. Follows from the above and the fact that localization and tensor commute. □

Corollary 18.4. *A module finitely presented M is flat if and only if it is locally free if and only if it is projective.*

19. 21/10/2019

19.1. **Cohen–Seidenberg theory.** Setup: A'/A integral extension.

Lemma 19.1. *Suppose A, A' are domains. Then A' is a field if and only if A is a field.*

Proof. For $x \in A$, $q/x \in A'$ is integral over A, and then we can use this to produce an inverse.

Conversely, is A is a field, we consider a minimal polynomial for any element $y \in A'$, and by inverting the constant term let us write an inverse of y. □

Definition 19.2. We say $\mathfrak{p}' \subseteq A'$ lies over $\mathfrak{p} \subseteq A$ if $\mathfrak{p} = \mathfrak{p}' \cap A$. Equivalently, if $\mathfrak{p}'$ maps to $\mathfrak{p}$ in $\operatorname{Spec} A' \to \operatorname{Spec} A$.

Theorem 19.3. *If $\mathfrak{p}'/\mathfrak{p}$, then $\mathfrak{p}'$ is maximal if and only if $\mathfrak{p}$ is maximal. If $\mathfrak{p}'_1, \mathfrak{p}'_2$ are two such distinct primes, no one is contained in the other. Given $\mathfrak{p}$ and $I \subseteq A'$ with $I \cap A \subseteq \mathfrak{p}$, then there is such $\mathfrak{p}'$ with $I \subseteq \mathfrak{p}'$.*

Proof. The first statement follows from the lemma for $A/\mathfrak{p} \subseteq A'/\mathfrak{p}'$.

Now localize at $S = A - \mathfrak{p} \subseteq A' - \mathfrak{p}'$. Then $\mathfrak{p}$ becomes maximal, and now it follows from the first statement.

Again localize to assume A is local with maximal ideal $\mathfrak{p}$. Then take a maximal ideal of the extension containing I. It must lie over $\mathfrak{p}$ since it is the only maximal ideal. □

Proposition 19.4. *Let $L = \mathrm{Frac} A', K = \mathrm{Frac} A$. Then $l \in L$ is in A' if and only if its minimal polynomial has coefficients in A.*

Theorem 19.5 (Going down theorem)**.** *Suppose A, A' are domains, with A normal. Given $\mathfrak{q}'/\mathfrak{q}$ and $\mathfrak{p} \subseteq \mathfrak{q}$, there is a $\mathfrak{p}'$ lying over $\mathfrak{p}$ and with $\mathfrak{p}' \subseteq \mathfrak{q}'$.*

20. 23/10/2019

Proof. Consider the minimal polynomial $F_{a'} \in K[x]$ of $a' \in K' := \mathrm{Frac} A'$. We can prove that if $a' \in A'$, the $F_{a'} \in A[x]$. If in addition $a' \in \mathfrak{p}A'$, then we can prove $F_{a'} \in x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_i \in \mathfrak{p}$.

Now consider $\mathfrak{p}A'_{\mathfrak{q}'} \cap A$. It contains $\mathfrak{p}$, but if a is an element of it not in $\mathfrak{p}$, we have $a = a'/s'$ with $a' \in \mathfrak{p}A'$ and $s' \in A' - \mathfrak{q}'$. Then $a^n F_{s'}(x) = F_{a'}(ax)$. By the above, this has coefficients in $\mathfrak{p}$, and so the same is true for $S_{s'}$. So $s' \in \mathfrak{p}$, which is a contradiction. Hence $\mathfrak{p}A'_{\mathfrak{q}'} \cap A = \mathfrak{p}$.

Now let $\mathfrak{p}''$ be a ideal maximal along the ones of $A'_{\mathfrak{q}'}$ containing $\mathfrak{p}A'_{\mathfrak{q}}$ and disjoint from $A - \mathfrak{p}$. This exists by the above. We can prove this is a prime ideal, and $\mathfrak{p}'' \cap A = \mathfrak{p}$. Now $\mathfrak{p}' = A' \cap \mathfrak{p}''$. □

Theorem 20.1 (Noether normalization)**.** *Let k be a field, A a finitely generated k-algebra, and $I_1 \subset I_2 \subset \cdots \subset I_r \subset A$ are ideals, then there are algebraically independent $t_1, \ldots, t_m \in A$ such that A is module-finite over $P := k[t_1, \ldots, t_m]$, and for each i, $I_t \cap P = k[t_1, \ldots, t_l]$.*

21. 25/10/2019

Missed

Proof and applications of Noether normalization.

Theorem 21.1. *If A is a fnitely generated k-algebra for a field k, then if A is a field, then A is a finite extension of k.*

22. 28/10/2019

Theorem 22.1. *If A is a finitely generated k-algebra, then* $\sqrt{0} = \operatorname{rad} A$.

Proof. Suppose $f \in \sqrt{0}$. Then $\operatorname{Spec} A_f$ has the points $\mathfrak{p} \not\ni f$, so $\operatorname{Spec} A_f \neq 0$, so $A_f \neq 0$. Choose a maximal ideal $\mathfrak{m}$ of A_f. Then look at $A \to A_f \to A_f/\mathfrak{m}$, and this get us an ideal $\mathfrak{m} \subseteq A$, and we have a map $A/\mathfrak{m} \hookrightarrow A_f/\mathfrak{m}$. Now A_f is finitely generated k-algebra, so $A_f/\mathfrak{m}$ is a finite extension of k by the previous theorem. Hence $A/\mathfrak{m}$ is a finitely generated k-vector space, hence integral over k. This implies that $A/\mathfrak{m}$ is a field. Now we can see that $f \notin \mathfrak{m}$. □

Theorem 22.2 (Hilbert Nullstelensatz)**.** *Let A be a finitely generated k-algebra, and I an ideal of A. Then*

$$\sqrt{I} = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}.$$

Proof. Apply the previous theorem to A/I. □

22.1. **Dimension theory.**

Definition 22.3. Given a field extension $L \supseteq k$, let T be a maximal algebraically independent subset of L over k. Then we denote $\operatorname{tr.deg}(L/k) = \#T$ the *transcendental degree of* L/k.

Example 22.4. If L is a finitely generated as a field extension over k, then $\operatorname{tr.deg}(L/k)$ is finite, and $L/k(T)$ is a finite extension.

Lemma 22.5. *Let A be finitely generated k-algebra which is a domain. Let $K = \text{Frac} A$. Let $d := \text{tr.deg}(K/k)$. Suppose we have a chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$. Then $\sup r = d$, and $r = d$ if and only if the chain is maximal.*

Proof. Let $T_i = (t_1, \ldots, t_d)$. Then we Noether normalization, $\mathfrak{p}_i \cap P$ are contained in the chain $T_0 \subset \cdots \subset T_d$, and they are distinct by incomparability. Hence $r \leq d$. Moreover, we will use the going up/down to extend the chain. We of course assume $\mathfrak{p}_0 = 0$ and $\mathfrak{p}_r$ maximal, which implies that $\mathfrak{p}_r \cap P = T_d$. Suppose i is the smallest index with $\mathfrak{p}_i \cap P \neq T_i$. Then apply going down to $A/\mathfrak{p}_{i-1}$ over P/T_{i-1}, since the bottom one is a domain (is a polynomial ring over k). □

Definition 22.6. For any $A \neq 0$ ring, we define its dimension as $\sup r$ for $\mathfrak{p}_0 \subset \cdots \mathfrak{p}_r \subseteq A$.

Theorem 22.7. *If A is a finitely generated k-algebra that is a domain, then for $\mathfrak{p} \in \text{Spec} A$, we have $\dim A_\mathfrak{p} + \dim A/\mathfrak{p} = \dim A$. In particular, if $\mathfrak{m}$ is a maximal ideal, $\dim A_\mathfrak{m} = \dim A$.*

23. 30/10/2019

Theorem 23.1. *Let A be a finitely generated k-algebra. Consider $\mathfrak{p} \subseteq \mathfrak{q}$ primes of A. Then every maximal chain of primes between them have the same length,*

Proof. We may assume $\mathfrak{p} = 0$, but then A is a domain, and we are done by the above. □

23.1. Noetherianess.

Definition 23.2. A commutative ring A is Noetherian if one of the following equivalent conditions hold.

(1) Every ideal is finitely generated.
(2) A satisfies the ascending chain condition ($I_1 \subseteq I_2 \subseteq \cdots$ stabilize).
(3) For any set of ideals S, there is a maximal element.

Definition 23.3. A commutative ring A is Artinian if one of the following equivalent conditions hold.

(1) A satisfies the descending chain condition ($I_1 \supseteq I_2 \supseteq \cdots$ stabilize).
(2) For any set of ideals S, there is a minimal element.

Proposition 23.4. *Quotients an localizations of Noetherian rings are Noetherian.*

Lemma 23.5. *If A is Noetherian, then so is $A[x]$.*

Proof. Let I be an ideal of $A[x]$. Let $I_n = \{f \in I \colon \deg f = n\}$ and $L_n = \{lc(f) \colon f \in I_n\} \cup \{0\}$. Then L_n is an ideal, and $L_{n-1} \subseteq L_n$. Since A is Noetherian, they stabilize to L. This proves that there is n such that I_n generate I. Now choose polynomials that generate each L_m for $m \leq n$. They generate $A[x]$. □

Theorem 23.6 (Hilbert basis)**.** *Let A be Noetherian, and B a finitely generated A-algebra. Then B is Noetherian.*

Proof. This follows from the two previous results. □

Definition 23.7. For a ring A and a module M, M is called Noetherian if one of the following equivalent statements hold.

(1) Every submodule of M is finitely generated.
(2) Satisfies the ascending chain condition.
(3) Every subset of submodules has a maximal element.

Remark 23.8. A is Noetherian if anf only if it is Noetherian as an A-module. One can also define an Artinian module.

Proposition 23.9. *Noetherianess and finitely-generatedness of modules is stable under extensions.*

Proposition 23.10. *Let A be Noetherian and M an A-module. Then M is Noetherian if and only if M is finitely generated if and only if it is finitely presented.*

Lemma 23.11. *Consider $A \subseteq B \subseteq C$ rings. Suppose A is Noetherian, C is a finitely generated A-algebra and C is integral over B. Then B is a finitely generated A-algebra.*

Proof. C is a finitely generated integral B-algebra, so C is finitely generated as a B-module. Write

$$C = A[c_1, \dots, c_m] = Bc'_1 + \cdots + Bc'_n,$$

and write $c_i = \sum_j b_{ij} c'_j$ and $c'_i c'_j = \sum_k b_{ijk} c'_k$.

Let $B_0 = A[b_{ij}, b_{ijk}]$ a finitely generated A-algebra. Then $C = B_0 + B_0 c'_1 + \cdots + B_0 c'_n$, since it contains c_i and is closed under multiplication by c_i. Now C is a Noetherian B_0-module, and B is a finitely generated B_0-module since it is a submodule of C. This implies B is a finitely generatd A-algebra. □

24. 31/10/2019

Theorem 24.1 (Noether's theorem on invariant subrings)**.** *Let k be a Noetherian ring, A a finitely generated k-algebra, and $G \subseteq \mathrm{Aut}_{k\text{-}alg} A$ a finite group. Then A^G is also a finitely generated algebra over k. (Slogan: finite quotients of varieties are varieties)*

Example 24.2. $A = k[x_1, \ldots, x_n]$, $G = S_n$, then $A^G \simeq k[y_1, \ldots, y_n]$.

Proof. We just need to check that A is integral over A^G: an element a is a root of $\prod_{g \in G}(x - ga)$. □

Lemma 24.3 (Noetherian induction)**.** *Let X be a Noetherian space, and $\mathscr{P}$ a property of closed subsets of X. Assume that whenever $\mathscr{P}$ holds for all proper $Z \subset Y$, then $\mathscr{P}$ holds for Y. Then $\mathscr{P}$ holds for any closed subset.*

Theorem 24.4 (Chevalley's theorem)**.** *Let $A \to B$ be a homomorphism between finitely generated k-algebras where k is Noetherian. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ maps constructible sets to constructible sets.*

24.1. **Associated primes.**

Definition 24.5. For a ring A and module M, a prime $\mathfrak{p}$ is an associated prime of M if it is $\mathfrak{p} = \mathrm{Ann}(m)$ for some $m \in M$. Equivalently, M contains a submodule isomorphic to $A/\mathfrak{p}$. We denote the set of associated primes by $\operatorname{Ass} M$. If I is an ideal, we denote $\operatorname{Ass} I := \operatorname{Ass}(A/I)$.

25. 04/11/2019

Proposition 25.1. *If $0 \to M_1 \to M_2 \to M_3 \to 0$, then $\operatorname{Ass} M_1 \subseteq \operatorname{Ass} M_2 \subseteq \operatorname{Ass} M_1 \cup \operatorname{Ass} M_3$.*

Proposition 25.2. *If A is Noetherian or M is Notherian, then*

$$\mathrm{z.div}\, M = \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}.$$

Proposition 25.3. $\operatorname{Ass} M \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} V(\mathfrak{p}) \subseteq \mathrm{Supp}\ M \subseteq V(\mathrm{Ann}\ M$. *We have $=$ in the third if A or M is Noetherian and in the fourth if M is finitely generated.*

Also, If A or M is Noetherian, then the minimal elements of $\mathrm{Supp}\ M$ *are associated primes.*

26. 06/11/2019

Lemma 26.1. *Let M be Noetherian. Then there is a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$ for some primes $\mathfrak{p}_i$. For any such chain, $\operatorname{Ass} M \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \operatorname{Supp} M$.*

Proof. If $M \neq 0$, then $\operatorname{Ass} M \neq \emptyset$, so we have $A/\mathfrak{p}_1 \hookrightarrow M$, so that is M_1, and then repeat the process to the quotient. This process terminates since M is Noetherian.

Now the second part of the claim follows from the statement about $\operatorname{Ass} M$ under exact sequences. □

Corollary 26.2. *If M is Noetherian, then $\operatorname{Ass} M$ is finite.*

Example 26.3 (Geometric interpretation)**.** Assume M is Noetherian finitely generated. Then $\operatorname{Supp} M$ is closed, and

$$\operatorname{Ass} M = \{\text{minimal assoc. primes}\} \sqcup \{\text{nonminimal assoc. primes}\}$$

so

$$\operatorname{Ass} M = \{\text{minimal primes containing Ann } M\} \sqcup \{\text{embedded primes}\}$$

which is

$$\operatorname{Ass} M = \{\text{irreducible components of } M\} \sqcup \{\text{embedded components}\}$$

26.1. **Primary decomposition.** For this entire section, A is Noetherian, and M is a finitely generated A-module.

Definition 26.4. A submodule $N \subseteq M$ is $\mathfrak{p}$-primary if $\operatorname{Ass} M/N = \{\mathfrak{p}\}$. We say it is primary if it is $\mathfrak{p}$-primary for some $\mathfrak{p}$.

Definition 26.5. M is coprimary if $\operatorname{Ass} M$ is a singleton. Equivalently, $0 \subseteq M$ is primary.

Proposition 26.6. *A finite intersection of $\mathfrak{p}$-primary submodules of M is $\mathfrak{p}$-primary.*

Proof. If N_1, N_2 are two such $\mathfrak{p}$-primary, then

$$0 \to M/(N_1 \cap N_2) \to M/N_1 \oplus M/N_2,$$

and hence $\operatorname{Ass} M/(N_1 \cap N_2) \subseteq \operatorname{Ass} M/N_1 \cup \operatorname{Ass} M/N_2 = \{\mathfrak{p}\}$. □

Proposition 26.7. *For $M \neq 0$, the following are equivalent. (i) M is $\mathfrak{p}$-coprimary, (ii) $\mathfrak{p}$ is minimal prime containing* $\operatorname{Ann} M$, *and* $\operatorname{z.div} M \subseteq \mathfrak{p}$ *(iii) $\mathfrak{p}^n M = 0$ for some $n \geq 1$ and* $\operatorname{z.div} M \subseteq \mathfrak{p}$.

Proof. (i) $\implies$ (ii): $\operatorname{Ass} M = \{\mathfrak{p}\}$, and we have proven that the set of miimal associated primes of M ($\mathfrak{p}$ in this case) are the minimal primes containing Ann M. The second part follows since all associated primes are contained in $\mathfrak{p}$.

(ii) $\implies$ (iii): The second part of (ii) means that $M \to M_{\mathfrak{p}}$ is injective. Now $\mathfrak{p}^n M \hookrightarrow \mathfrak{p}^n M_{\mathfrak{p}}$. So we mat reduce to the case $A = A_{\mathfrak{p}}$ and $M = M_{\mathfrak{p}}$, in which case $\mathfrak{p}$ is the maximal ideal. Since $\mathfrak{p}$ is maximal and is minimal containing Ann M, this means $\sqrt{\text{Ann } M} = \mathfrak{p}$. Since $\mathfrak{p}$ is finitely generated, $\mathfrak{p}^n M = 0$ for some n.

(iii) $\implies$ (i): $\mathfrak{p}^n \subseteq \text{Ann } M \subseteq \operatorname{z.div} M \subseteq \mathfrak{p}$. Now for any $\mathfrak{q} \supseteq \text{Ann } M$, we have $\mathfrak{p} \subseteq \mathfrak{q}$. This proves that $\mathfrak{p}$ is a minimal prime containing Ann M, which means it is a minimal associated prime. Since the second part of (iii) means every associated prime is contained in $\mathfrak{p}$, this concludes the argument. □

27. 08/11/2019

Recall we are assuming that A is Noetherian ad M is a finitely generated A-module.

Example 27.1. $I \subseteq A$ is $\mathfrak{p}$-primary if and only if $\mathfrak{p}^n \subseteq A$ and for all $a, b \in A$, $ab \in I \implies a \in \mathfrak{p}$ or $b \in I$.

Corollary 27.2. *If M is $\mathfrak{p}$-coprimary, then $M \hookrightarrow M_{\mathfrak{p}}$. If M is $\mathfrak{q}$-coprimary for some $\mathfrak{q} \not\subseteq \mathfrak{p}$, then $M_{\mathfrak{p}} = 0$.*

Theorem 27.3 (Primary decomposition)**.** *Let A be Noetherian an M a finitely generated A-module. Let $N \subseteq M$ be a submodule. Then there is a decomposition $N = \bigcap_{\mathfrak{p} \in \operatorname{Ass} M/N} N^{\mathfrak{p}}$ where each $N^{\mathfrak{p}} \subseteq M$ is $\mathfrak{p}$-primary. If $\mathfrak{p}$ is minimal in $\operatorname{Ass} M/N$, then $N^{\mathfrak{p}} = \ker\left(M \to (M/N)_{\mathfrak{p}}\right)$. Moreover, this decomposition commutes with localization.*

For the embedded primes, $N^{\mathfrak{p}}$ is not necessarily uniquely determined.

Proof. We say a submodule is irreducible if it is not the intersection of two strictly larger submodules.

By Noetherianess, every submodule can be written as a finite intersection of irreducible submodules.

Now it suffices to prove an irreducible submodule if primary. By quotienting, we may assume 0 is irreducible in M, and need to prove M is coprimary. If not, then there are at least two associated primes $\mathfrak{p}_1, \mathfrak{p}_2$, but then we have submodules $M_1 \simeq A/\mathfrak{p}_1$, $M_2 \simeq A/\mathfrak{p}_2$ of M. Now $m \in M_1, M_2$

would have $\mathfrak{p}_1 = \mathrm{Ann}(m) = \mathfrak{p}_2$ if $m \neq 0$, and this is not the case. Hence $M_1 \cap M_2 = 0$, and this is a contradiction.

Now we have a decomposition $N = \bigcap_i P_i$ with $\mathrm{Ass}\, M/N \subseteq \{\mathfrak{p}_i\}$, since $M/N \hookrightarrow \bigoplus_i M/P_i$. We can remove redundant factors, and then $\mathrm{Ass}\, M/M = \{\mathfrak{p}_i\}$. Let $N' = \bigcap_{i \neq j} P_i$, so that $N = N' \cap P_j$, and so $0 \neq N'/N \hookrightarrow M/P_j$ and $N'/N \hookrightarrow M/N$, and so $\mathrm{Ass}\, N'/N = \{\mathfrak{p}_i\}$ is contained in $\mathrm{Ass}\, M/N$.

To prove the uniqueness at minimal primes, consider

$$\begin{array}{ccc} M & \longrightarrow & \left(\frac{M}{N}\right)_{\mathfrak{p}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \gamma} \\ \frac{M}{N^{\mathfrak{p}}} & \xrightarrow{\ \delta\ } & \left(\frac{M}{N^{\mathfrak{p}}}\right)_{\mathfrak{p}} \end{array}$$

δ is injective since $M/N^{\mathfrak{p}}$ is $\mathfrak{p}$-coprimary. From the injection $M/N \hookrightarrow \bigoplus M/N^{\mathfrak{q}}$, we get the map γ when we localize at $\mathfrak{p}$, since $\mathfrak{p}$ is a minimal prime in $\mathrm{Ass}\, M/N$. Now the claim follows from a Snake lemma in the diagram above. □

28. 13/11/2019

Theorem 28.1 (Krull intersection theorem). *Let A be a Noetherian domain, and $I \subset A$ an ideal. Then*

$$\bigcap_{n \geq 0} I^n = 0.$$

More generally, if A is a ring and M a Noetherian module, $I \subset A$ an ideal. Let $N = \bigcap I^n M$. Then there is $x \in I$ such that $(1 + x)N = 0$.

Proof. Choose a primary decomposition $IN = \bigcap_{\mathfrak{p}} Q^{\mathfrak{p}}$. We will prove that $N \subseteq Q^{\mathfrak{p}}$ for each $\mathfrak{p}$. If $I \subseteq \mathfrak{p}$, then $I^n(M/Q^{\mathfrak{p}}) = 0$ for n large enough, and so $N \subseteq I^n M \subseteq Q^{\mathfrak{p}}$. When $I \not\subseteq \mathfrak{p}$, choose $a \in I \setminus \mathfrak{p}$, and $aN \subseteq IN \subseteq Q^{\mathfrak{p}}$, but a is not a zerodivisor on $M/Q^{\mathfrak{p}}$, so $N \subseteq Q^{\mathfrak{p}}$. Hence $IN = N$, and then the same proof as Nakayama gives the result. □

28.1. **Jordan–Hölder filtration.**

Definition 28.2. Let A be a ring and M a module. We call M *simple* if $M \neq 0$ and does not have a smaller nontrivial submodule. This is equivalent to $M \simeq A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$.

Definition 28.3. For a decreasing filtration $M_\bullet$, we let the associated graded module $\mathrm{gr}_\bullet M$ be $\mathrm{gr}_i(M) = M_i/M_{i+1}$. $M_\bullet$ is a *Jordan–Hölder filtration* or *composition series* if each quotient is simple.

29. 15/11/2019

Lemma 29.1. *Given* $0 \to N \to M \to Q \to 0$ *and a filtration of* $M_\bullet$, *this induces a filtration on* N *and* Q *such that* $0 \to N_\bullet \to M_\bullet \to Q_\bullet \to 0$. *This also implies* $0 \to \mathrm{gr}_\bullet(N) \to \mathrm{gr}_\bullet(M) \to \mathrm{gr}_\bullet(Q) \to 0$.

Theorem 29.2 (Jordan–Hölder)**.** *If a module* M *has a Jordan–Hölder filtration, then the multiset* $\{\mathrm{gr}_i(M)\}$ *is independent of the choice of such filtration. Any filtation can be refined to a Jordan–Hölder filtration. Moreover,* $\mathrm{Supp}(M) = \{$*maximal ideals* $\mathfrak{m} \colon A/\mathfrak{m} \simeq \mathrm{gr}_i(M)\}$, *and* $M \xrightarrow{\sim} \bigoplus_{\mathfrak{m} \in \mathrm{Supp}(M)} M_{\mathfrak{m}}$.

Proof. Note that if $M_\bullet$ is a Jordan–Hölder filtration, and any $0 \to N \to M \to Q \to 0$, then $\{\mathrm{gr}_i(M)\} = \{\mathrm{gr}_i(N)\} \cup \{\mathrm{gr}_i(Q)\}$ as multisets, and so if $M \supset N = N_1 \supset N_2 \supset \cdots 0$ is another filtration, then this reduced the problem to N_1, and then do the same thing.

The last two claims are clear by localizing. □

Definition 29.3. The length $l(M)$ of a module M is the length of a Jordan–Hölder filtration, or ∞ if it does not exist.

Lemma 29.4. *M is Artinian and Noetherian if and only if* $l(M) < \infty$.

Proof. Since M is Noetherian, we can find maximal proper submodules. This gives a filtration, and it is finite since M is Artinian.

Conversely, any filtration refines to a Jordan–Hölder filtration, so must be finite. □

Theorem 29.5. *A is Artinian if and only if A is Noetherian and* $\dim A = 0$.

30. 18/11/2019

Proof. If A is Noetherian and $\dim A = 0$, then we proved there is a finite filtration $A_\bullet$ with $\mathrm{gr}_\bullet$ being the quotient of a prime. But $\dim A = 0$, so all of these are fields, and hence are simple. So $l(A) < \infty$, hence A is Artinian.

$\Longrightarrow$: Suppose A is Artinian. Let $\mathscr{P}$ be the set of finite products of maximal ideals. Since A is Artinian, we can find $\mathfrak{p} \in \mathscr{P}$ a minimal element. In particular, this means that for any maximal ideal $\mathfrak{m}$, we have $\mathfrak{m}\mathfrak{p} = \mathfrak{p}$. In particular, $\mathfrak{p} \subseteq \mathrm{rad}\, A$. Now let $\mathscr{I} = \{I \subseteq \mathfrak{p} \colon I\mathfrak{p} \neq 0.\}$

If $\mathfrak{p} \neq 0$, then $\mathfrak{p} \in \mathscr{I}$, so $\mathscr{I} \neq \emptyset$, so we can find $I \in \mathscr{I}$ minimal. Then I is principal, since if $a \in I$ is with $A\mathfrak{p} \neq 0$, then $(a) \in \mathscr{I}$, so $I = (a)$. Now also $(I\mathfrak{p})\mathfrak{p} = I\mathfrak{p} \neq 0$, so $I\mathfrak{p} \in \mathscr{I}$. This implies $I\mathfrak{p} = I$. Hence by Nakayama, we have $I = 0$. This is a contradiction.

Hence $\mathfrak{p} = 0$, that is, there is a product of maximal ideals which is 0. Such product also gives us a Jordan–Hölder filtration: If $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$, then $(\mathfrak{m}_1 \cdots \mathfrak{m}_i)/(\mathfrak{m}_1 \cdots \mathfrak{m}_{i+1})$ is an Artinian $A/\mathfrak{m}_{i+1}$-module, hence a finite dimensional vector-space, and so has finite length. This proves that A has finite length, hence A is Noetherian.

Also, $\{\text{primes}\} = \operatorname{Supp} A \subseteq \{\mathfrak{m}_i\}$, so A has dimension 0. □

Corollary 30.1. *If A is Artinian, then $A = A_1 \times \cdots \times A_n$ for some local Artinian rings A_i.*

Proposition 30.2. *If A is a local Artinian ring, then the maximal ideal is nilpotent.*

Proof. Since A is Artinian, $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. for some n, and since A is Noetherian, $\mathfrak{m}^n$ is finitely generated, so by Nakayama we have $\mathfrak{m}^n = 0$. □

30.1. **Graded rings.**

Lemma 30.3. *Let M be a graded A module, with A finitely generated A_0-algebra, M a finitely generated A-module. Then each M_n is finitely generated A_0-module, and $M_n = 0$ for $n << 0$.*

Proof. Let $A = A_0[a_1, \ldots, a_r]$ with a_i homogeneous, and $M = \sum_{i=1}^{s} Am_i$ with m_i homogeneous. Now $M = \sum_{i,j} A_0 a_i m_j$ as an A_0-module, and so the claim follows. □

31. 20/11/2019

Definition 31.1. Consider the graded ring $A = k[x_0, \ldots, x_r]$ and M a finitely generated graded A-module. We defined the *Hilbert function of* M by $n \mapsto \dim_k M_n$. We define the Hilbert series by $H_M(t) = \sum_{n \in \mathbb{Z}} (\dim M_n) t^n$.

Example 31.2. $H_A(t) = \frac{1}{(1-t)^{r+1}}$

Theorem 31.3 (Hilbert–Serre). $H_M(t) = \frac{f(t)}{(1-t)^{r+1}}$ *for some* $f(t) \in \mathbb{Z}[t, t^{-1}]$.

Corollary 31.4. *There exist a polynomial $h_M \in \mathbb{Q}[t]$ such that $h_m(t) = \dim M_n$ for $n >> 0$.*

Proof of Theorem. Do induction on r. For $r = -1$ this is trivial. Consider the homomorphism of graded modules $M(-1) \xrightarrow{\cdot x_0} M$. Now the kernel and cokernel are finitely generated graded $A/(x_0)$-modules, and one can do the induction. □

Theorem 31.5. *Let A a graded ring, M a finitely generated graded A-module, A_0 Artinian and A generated by homogeneous elements x_i of degree k_i. Then if $H_M(t) = \sum_{n\in\mathbb{Z}} l(M_n)t^n$, then $H_M(t) = \frac{f(t)}{\prod_i(1-t^{k_i})}$*

31.1. **Filtered modules.** Let A be a Noetherian ring and $\mathfrak{q}$ an ideal. We consider filtrations compatible with $\mathfrak{q}$, in the sense that $\mathfrak{q}M_i \subseteq M_{i+1}$.

We consider the function $n \mapsto l(M/M_n)$. We will prove it eventually is a polynomial.

Definition 31.6. We form $\overline{A} = A \oplus \mathfrak{q} \oplus \mathfrak{q}^2 \oplus \cdots$ and $\overline{M} = M_0 \oplus M_1 \oplus \cdots$. So $\overline{A}$ is a graded ring and $\overline{M}$ is a graded $\overline{A}$-module.

32. 22/11/2019

Lemma 32.1. *$\overline{A}$ is Noetherian.*

Proof. Hilbert basis theorem: it is generated as an A-algebra by (finitely many) generators of $\mathfrak{q}$. □

Proposition 32.2. *The following are equivalent: (1) $M_{n+1} = \mathfrak{q}M_n$ for $n >> 0$, (2) $\overline{M}$ is a finitely generated $\overline{A}$-module.*

In such case, we call the filtration $\mathfrak{q}$-stable.

Proof. (1) $\implies$ (2) is clear, since each M_i is finitely generated.

For the converse, choose homogeneous generators z_i of degree d_i. Then $M_n = \sum_i \mathfrak{q}^{n-d_i} z_i$, and so for $n > d_i$ for all i we have $M_{n+1} = \mathfrak{q}M_n$. □

Lemma 32.3 (Artin–Rees)**.** *Let $M_\bullet$ be a $\mathfrak{q}$-adic filtered module that if finitely generated over a Noetherian A. For $N \subseteq M$, we consider the induced filtration. If $M_\bullet$ is stable, then $N_\bullet$ also is.*

Proof. $\overline{N} \subseteq \overline{M}$, and since $\overline{A}$ is Noetherian, the claim follows. □

Theorem 32.4 (Samuel's theorem)**.** *Let A be a Noetherian ring, $M_\bullet$ a finitely generated $\mathfrak{q}$-filtered A-module. Suppose $l(M/\mathfrak{q}M) < \infty$ and $M_\bullet$ is $\mathfrak{q}$-stable. Then there exist a polynomial (Hilbert–Samuel polynomial) $P_{M_\bullet}(t) \in \mathbb{Q}[t]$ such that $l(M/M_n) = P_{M_\bullet}(n)$ for $n \gg 0$. Moreover, the leading term only depends on $M, \mathfrak{q}$.*

Proof. Note $l(M/\mathfrak{q}M) < \infty$ iff $\operatorname{Supp} M \cap V(\mathfrak{q})$ consist of maximal ideals.

Replace A by $A/(\text{Ann } M)$, so that we may assume $\text{Ann } M = 0$. Then this means $\operatorname{Supp} M = \operatorname{Spec} A$, and so $\operatorname{Spec} A/\mathfrak{q} = V(\mathfrak{q})$ consist only of maximal ideals. This means $A/\mathfrak{q}$ is Artinian.

$\mathrm{gr}_\bullet(A)$ is finitely generated by elements of degree 1, and $\mathrm{gr}_\bullet(M)$ is a finitely generated $\mathrm{gr}_\bullet(A)$-module (as $M_\bullet$ is $\mathfrak{q}$-stable). Now apply Hilbert–Serre.

The claim about the leading term follows from the fact that $\mathfrak{q}^n M \subseteq M_n \subseteq \mathfrak{q}^{n-m} M$ for some fixed m. □

33. 25/11/2019

Definition 33.1. For $d \geq \deg P_M$, write $p_M(n) = e(d, M)\frac{n^d}{d!} + \cdots$. We call $e(d, M) \in \mathbb{Z}$ the multiplicity.

Proposition 33.2. *If* $0 \to N \to M \to Q \to 0$, *then* $\deg p_N \leq \deg p_M$ *and* $p_M \equiv p_N + p_Q \mod x^{\deg p_N - 1}$.

Proof. If $M_\bullet$ is any $\mathfrak{q}$-filtration, then p is additive if we take the induced filtrations on M. Now the claim follows from Artin–Rees. □

Corollary 33.3. *If* $d \geq \deg p_M$, *then* $e(d, M) = e(d, N) + e(d, Q)$.

33.1. **Dimension theory.**

Definition 33.4. For a module M, we define $\dim M = \sup\{r : \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r\}$ for primes $\mathfrak{p}_i \in \operatorname{Supp} M$.

Proposition 33.5. *If* M *is Noetherian, then* $\dim M = \max\{\dim A/\mathfrak{p} : \mathfrak{p}$ *is minimal among primes in* $\operatorname{Supp} M\}$.

Proof. There are finitely many minimals elements of $\operatorname{Supp} M$, and every $\mathfrak{q} \in \operatorname{Supp} M$ contains such a minimal element. □

Assume from now on that A is Noetherian local, and $\mathfrak{q}$ be $\mathfrak{m}$ primary (same as $\mathfrak{q} \supseteq \mathfrak{m}^n$, same as $l(A/\mathfrak{q}) < \infty$ and $\mathfrak{q} \neq 0$).

Let M be a nonzero finitely generated module.

Then $d(M) := \deg p_M$ does not depends on $\mathfrak{q}$.

Definition 33.6. Let $s(M)$ be the smallest s such that there are $x_1, \ldots, x_s \in \mathfrak{m}$ with $l(M/(x_1, \ldots, x_s)) < \infty$. These are called a system of parameters.

Remark 33.7. This correspond to a local "chart" that is finite-to-1.

Theorem 33.8 (Dimension theorem)**.** $\dim M = d(M) = s(M)$.

Lemma 33.9. *If $x \in \mathfrak{m}$, then $s(M) \le s(M/xM) + 1$. If x does not vanish on any irreducible component of* $\operatorname{Supp} M$ *of maximal dimension, then* $\dim M/xM + 1 \le \dim M$*. If x is not a zerodivisor on M, then $d(M/xM) \le d(M) - 1$.*

Proof. The first two are trivial. For the last, we have $0 \to M \xrightarrow{x} M \to M/xM$, and so $d(M/xM) < d(M)$. □

34. 27/11/2019

Proposition 34.1. $\dim M \le d(M)$.

Proof. Base case: $d(M) = 0$, in which case $\mathfrak{m}^n M$ stabilize. By Nakayama, we have $\mathfrak{m}^n M = 0$. Now $\operatorname{Supp} M = V(\operatorname{Ann} M) \subseteq V(\mathfrak{m}^n) = \{\mathfrak{m}\}$. Hence $\dim M = 0$.

Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$ be a maximal chain in $\operatorname{Supp} M$. Then $\dim M = \dim A/\mathfrak{p}_0$. $\mathfrak{p}_0$ is a minimal prime, so $\mathfrak{p}_0$ is an associated prime and let $N \subseteq M$ isomorphic to $A/\mathfrak{p}_0$. Then $d(N) \le d(M)$, and so it suffices to prove $r \le d(N)$. Choose $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_0$. Then $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r \subset N/xN$. So $d-1 \le \dim N/xN \le d(N/xN)$ by induction. And then by the lemma we have $d(N/xN) \le d(N)-1$ since x is not a zerodivisor. □

Proposition 34.2. $d(M) \le s(M)$.

Proof. Let $x_1, \dots, x_s$ be a minimal system of parameters. Since $(x_1, \dots, x_s) \subseteq \mathfrak{m}$, we have $d(M) \le d((x_1, \dots, x_s)^\bullet M)$. By Samuel's theorem, this is at most s. □

Proposition 34.3. $s(M) \le \dim M$.

Proof. Base case $\dim M = 0$: then $\operatorname{Supp} M = \{\mathfrak{m}\}$, so M has finite length.

Consider $\{\mathfrak{p} \in \operatorname{Supp} M \colon \dim A/\mathfrak{p} = \dim M\}$ which are contained in the minimal primes of $\operatorname{Supp} M$, which is a finite set. $\mathfrak{m}$ is not in such set if $\dim M > 0$. Choose $x \in \mathfrak{m}$ outside such set. By the lemma, we have $\dim M/xM \le \dim M - 1$. Also by the lemma, $s(M) \le \ s(M/xM)$, and we are done. □

Corollary 34.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field k. Then* $\dim \mathfrak{m}/\mathfrak{m}^2 \ge \dim A$.

Proof. By Nakayama we have that $\dim \mathfrak{m}/\mathfrak{m}^2$ is the minimal number of generators of $\mathfrak{m}$, which serves as a system of parameters, hence is at least $s(A)$. □

34.1. **Height.**

Definition 34.5. Let $\mathfrak{p}$ be a prime of a ring A. Then its height $h(\mathfrak{p})$ is $\dim A_{\mathfrak{p}}$.

Theorem 34.6. *Let A be a Noetherian ring and $f_1, \dots, f_r \in A$. Let $\mathfrak{p}$ be a prime minimal among primes that contain $(f_1, \dots, f_r)$. Then $h(\mathfrak{p}) \leq r$.*

Proof. By the assumptions, the only prime ideal in $A_{\mathfrak{p}}/(f_1, \dots, f_r)$ is $\mathfrak{p}$. Hence $r \geq s(A_{\mathfrak{p}}) = h(\mathfrak{p})$. □

35. 02/12/2019

Krull's principal ideal theorem, regularity, inverse limits

36. 04/12/2019

Theorem 36.1. *Given a short exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$, then the inverse limit is left exact, and is right exact if $A_\bullet$ has only surjective maps.*

Proof. Think of the inverse limit as the kernel of $\Theta_A \colon \prod A_n \to \prod A_n$ given by $(x_i) \mapsto (x_i - \mathrm{im}(x_{i+1}))$, and apply Snake lemma. □

Definition 36.2. For a ring A with an ideal I, define the completion $\hat{A} := \varprojlim_n A/I^n$. For a module M, $\hat{M} := \varprojlim_n M/I^n$.

Proposition 36.3. *For a topological abelian group, $M \to \hat{M}$ is injective if and only if M is separated (i.e. Hausdorff).*

37. 06/12/2019

Theorem 37.1. *Let A be a local ring. Then $\hat{A}$ is flat A-module, we have $A/\mathfrak{m}^n \simeq \hat{A}/\mathfrak{m}^n$. $\hat{A}$ is complete, and $\mathfrak{m} \subseteq \hat{A}$.*

38. 09/12/2019

Theorem 38.1. *If A is a Noetherian local ring, then so is $\hat{A}$. They have the same dimension, and $\hat{A}$ is regular if and only if A is.*

Proof. We clearly have that $\hat{\mathfrak{m}}$ is maximal. But it is contained in the radical, and so $\hat{\mathfrak{m}}$ is the unique maximal ideal. Since $\mathrm{gr}_\bullet A \simeq \mathrm{gr}_\bullet \hat{A}$, the right one must be Noetherian. Now let I be an ideal of $\hat{A}$, with the induced filtration $\mathrm{Fil}_\bullet I$. Then $\mathrm{gr}_\bullet I$ is an ideal of a Noetherian ring, so is finitely generated. Take homogeneous generators for it, and take lifts. Then using that $\hat{A}$ is complete and the Krull intersection theorem, we can prove they generate I.

By the above, the quotients of the filtration are isomorphic, and so the Hilbert–Samuel polynomials are the same, so $\dim \hat{A} = \dim A$.

Regularity also follows since $\mathfrak{m}/\mathfrak{m}^2 \simeq \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2$. □

38.1. DVRs.

Definition 38.2. A *discrete valuation ring* is the ring of integers of a discrete valued field.

One can check that for t a uniformizer, that $\mathfrak{m} = (t)$, and so the ideals are only $\mathfrak{m}^e = (t^e)$. Hence it is regular of dimension 1.

39. 11/12/2019

39.1. Depth.

Definition 39.1. For a ring A and a module M. Consider sequences $x_1, \dots, x_n \in A$ and let $M_i = M/(x_1, \dots, x_i)M$. x_i is a M-regular sequence if for all i, $x_i \notin \mathrm{z.div}(M_{i-1})$.

Remark 39.2. If M is Noetherian, this means all components of M_{i-1} (including embedded) get cut out.

Definition 39.3. For an ideal $\mathfrak{a}$ with $\mathfrak{a}M \subset M$, let $\mathrm{depth}_{\mathfrak{a}} M$ be the supremum of M-regular sequences contained in $\mathfrak{a}$.

Example 39.4. Let $A = k[x,y]_{(x,y)}$ and $M = A/(xy, y^2)$. Then $\mathrm{depth}\, M = 0$ since (x,y) is an embedded component.

Let P_1, P_2 be two $\mathbb{A}^2$ inside A^4 meeting transversely. Then any attempt at cutting will still have an embedded prime.

Assume for simplicity that A is local Noetherian and M is finitely generated.

Proposition 39.5. $\operatorname{depth} M = 0$ *if an only if* $\mathfrak{a} \in \operatorname{Ass} M$, *and* $\operatorname{depth} M \leq \dim M$. *If equality holds, we call* M Cohen–Macaulay.

Theorem 39.6. *A is a DVR if and only if it is (1) normal domain of dimension* 1, *(2) normal domain of depth* 1, *(3) regular ring of dimension* 1, *(4)* $\mathfrak{m}$ *is pricipal and* $\operatorname{height}(\mathfrak{m}) \geq 1$.