FALL 2022 LEARNING SEMINAR: IWASAWA THEORY

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November 21: Introduction to Iwasawa theory

1. What is Iwasawa theory?

1.1. **Inspiration from function fields.** Let X be a smooth projective variety over $\mathbb{F}_p$. Its *zeta function* was originally defined as

$$\zeta(X,s) = \exp\left(\sum_{m\geq 1}\frac{\#X(\mathbb{F}_{p^m})}{m}p^{-ms}\right).$$

To prove the easy parts of the Weil conjecture, one writes $X(\mathbb{F}_{p^m}) = X(\overline{\mathbb{F}}_p)^{\mathrm{Frob}^m=1}$, and rewrites this using Grothendieck–Lefschetz as

$$\#X(\mathbb{F}_{p^m}) = \sum_{k\geq 0}(-1)^k \mathrm{tr}\left(\mathrm{Frob}^{m,*} \mid H^k_{\text{ét}}(X_{\overline{\mathbb{F}_p}},\mathbb{Q}_l)\right)$$

and thus

$$\zeta(X,s) = \exp\left(\sum_{k\geq 0}(-1)^k \mathrm{tr}\left(\sum_{m\geq 1}\frac{1}{m}\mathrm{Frob}^m p^{-ms} \mid H^k_{\text{ét}}(X_{\overline{\mathbb{F}_p}},\mathbb{Q}_l)\right)\right) = \prod_{k\geq 0}\det\left(1-\mathrm{Frob}\cdot p^{-s} \mid H^k_{\text{ét}}(X_{\overline{\mathbb{F}_p}},\mathbb{Q}_l)\right)^{(-1)^{k+1}}.$$

That is,

$$\zeta(X,s) = \prod_{k\geq 0}\mathrm{char}\left(\mathrm{Frob}\cdot T \mid H^k_{\text{ét}}(X_{\overline{\mathbb{F}_p}},\mathbb{Q}_l)\right)^{(-1)^{k+1}}|_{T=p^{-s}}.$$

So, very roughly, we see some *extra structure* when we look at all the $\#X(\mathbb{F}_{p^m})$ together. For example, the $\#X(\mathbb{F}_{p^m})$ must satisfy a recurrence relation!

1.2. **Iwasawa's idea.** Now imagine we want to replace the variety X above by a number field F. Instead of $X(\mathbb{F}_p)$, we should have some other interesting arithmetic quantity. Iwasawa's original investigations were about $\mathrm{Cl}(F)$, so let's take that as the analogue.

For $X(\mathbb{F}_{p^m})$, we can think of this as the rational points of $X_{\mathbb{F}_{p^m}}$. If X corresponds to F, maybe $X_{\mathbb{F}_{p^m}}$ corresponds to a finite extension F_n of F. But what should be this tower of number fields? The Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. It turns out it will be easier, instead, to focus on one of the $\mathbb{Z}_p$ components.

Definition 1.1. A $\mathbb{Z}_p^d$-extension of a number field F is an infinite Galois extension F_∞ with $\mathrm{Gal}(F_\infty/F)\simeq\mathbb{Z}_p^d$. Concretely, this is a tower of number fields F_n where $\mathrm{Gal}(F_n/F)\simeq(\mathbb{Z}/p^n\mathbb{Z})^d$.

Remark 1.2. Leopoldt's conjecture for a number field F and a prime p is equivalent to[1]: if d is the largest positive integer such that there exist a $\mathbb{Z}_p^d$ extension of F, then $d = 1 + r_2(F)$, where $r_2(F)$ is the number of complex places of F. Leopoldt's conjecture is known for abelian extensions of $\mathbb{Q}$ and abelian extensions of a quadratic imaginary field.

Example 1.3. If $F = \mathbb{Q}$, there is a unique $\mathbb{Z}_p$-extension, contained inside the tower of cyclotomic fields $\mathbb{Q}(\mu_{p^n})$.

Example 1.4. If $F = K$ is a quadratic imaginary field, There is a unique $\mathbb{Z}_p^2$-extension K_∞. Complex conjugation acts on $\mathrm{Gal}(K_\infty/K)$, with eigencomponents $\mathrm{Gal}(K_\infty^{cycl}/K)$ and $\mathrm{Gal}(K_\infty^{anti}/K)$. Of course, K_∞^{cycl} is contained in the tower $K(\mu_{p^n})$. K_∞^{anti} is the unique $\mathbb{Z}_p$-extension contained in the tower of ring class fields of p-power conductor of K.

For concreteness, let's focus our attention on the cyclotomic $\mathbb{Z}_p$-extension. It is contained inside the tower $K_n := \mathbb{Q}(\mu_{p^{n+1}})$, say $F_n \subseteq K_n$ for $n \geq 0$. So $F_0 = \mathbb{Q}$ and $K_0 = \mathbb{Q}(\mu_p)$.

If we want an analogue of the zeta function ζ_X, we need to somehow assemble the groups $\mathrm{Cl}(K_n)$ together. It turns out that the groups $\mathrm{Cl}(K_n)$ do not behave well in families, but their p-primary parts do. So denote

$$X_n := \mathrm{Cl}(K_n)[p^\infty].$$

This is a $\mathbb{Z}_p[\mathrm{Gal}(K_n/\mathbb{Q})]$-module.

Definition 1.5. We let $X_\infty := \varprojlim_n X_n$ with transition maps given by the norm map $\mathrm{Nm}_{K_{n+1}/K_n} : \mathrm{Cl}(K_{n+1})[p^\infty] \to \mathrm{Cl}(K_n)[p^\infty]$. This is a $\mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/\mathbb{Q})]\!]$-module. Call $\Lambda^{cycl} := \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/\mathbb{Q})]\!]$.

Now note that

$$\mathrm{Gal}(K_\infty/\mathbb{Q}) = \varprojlim_n \mathrm{Gal}(K_n/\mathbb{Q}) = \varprojlim_n (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times = \mathbb{Z}_p^\times.$$

Assuming $p > 2$ for simplicity, we can choose a topological generator $\gamma \in (1+p\mathbb{Z}_p)^\times \xrightarrow[\sim]{\log} \mathbb{Z}_p$ (for example $\gamma = 1+p$), we identify

$$\mathrm{Gal}(K_\infty/\mathbb{Q}) = \Delta \times \mathbb{Z}_p$$

where $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times \overset{\omega}{\hookrightarrow} \mathbb{Z}_p^\times$ for ω the Teichmüller character.

Definition 1.6. Let $\Lambda := \mathbb{Z}_p[\![T]\!]$ denote the *Iwasawa algebra*. It is a complete regular local ring of dimension 2 with maximal ideal $\mathfrak{m} = (p, T)$.

Proposition 1.7. $\Lambda^{cycl} \simeq \Lambda[\Delta]$ *where* $T \in \Lambda$ *is identified with* $\gamma - 1$.[2]

Proof. We just need to show that $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/\mathbb{Q})]\!] \simeq \Lambda$. The problem is seeing that the map and its inverse are well-defined and continuous. That is, we need to see that

$$(T+1)^{p^n} \to 1 \text{ in } \Lambda$$

[1]This is explained in [Was97, Theorem 13.4]

[2]In general, the completed group algebra of a $\mathbb{Z}_p^d$ extension is identified with $\mathbb{Z}_p[\![T_1, \ldots, T_d]\!]$ in a similar way.

and that

$$(\gamma-1)^n \to 0 \text{ in } \mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/\mathbb{Q})]\!].$$

The first one simply follows from $(T+1)^{p^n}-1 \in \mathfrak{m}^{\min_{1\le a\le p^n}(a+\nu_p\binom{p^n}{a})}$. Since $\nu_p\binom{p^n}{a} = n-\nu_p(a)$ for $1\le a\le p^n$, we have $(T+1)^{p^n}-1\in\mathfrak{m}^{n+1}$.

For the second one, we need to show that for any $m\ge 0$, we have $(\gamma-1)^n \mod (\gamma^{p^m}-1)$ goes to 0 in $\mathbb{Z}_p[\mathrm{Gal}(F_m/\mathbb{Q})]$. Write $n=a_0+a_1p+\cdots+a_kp^k$ in base p. Then

$$(\gamma-1)^n = \prod_{i=0}^{k}(\gamma^{p^i}-1+p^i(\cdots))^{a_i} \equiv \prod_{i=0}^{m-1}(\gamma^{p^i}-1+p^i(\cdots))^{a_i}\cdot\prod_{i=m}^{k}(p^i(\cdots))^{a_i}.$$

So $(\gamma-1)^n \mod (\gamma^{p^m}-1)$ is divisible by $p^{\sum_{i\ge m} ia_i}$, and $\sum_{i\ge m} ia_i\to\infty$ as $n\to\infty$. □

Roughly speaking, the goal of Iwasawa theory in this case is to:

(1) Understand the structure of X_∞ as a $\Lambda^{cycl}=\Lambda[\Delta]$-module.
(2) "Descend" this information to the finite level modules X_n.

2. The Iwasawa algebra

[3] We can think of $\Lambda=\mathbb{Z}_p[\![T]\!]$ as the ring of functions of the closed p-adic unit disk. Such a function can only have finitely many zeroes, that is, we have:

Theorem 2.1 (p-adic Weierstraß preparation)**.** *Any element $f(T)\in\Lambda$ can be uniquely written as*

$$f(T)=p^\mu\lambda(T)u(T)$$

where $\mu\ge 0$, $u(T)\in\Lambda^\times$ and $\lambda(T)\in\mathbb{Z}_p[T]$ is a distinguished polynomial*, i.e. of the form*

$$\lambda(T)=T^n+a_{n-1}T^{n-1}+\cdots+a_1T+a_0 \quad \textit{where} \quad p\mid a_i.$$

We call μ the μ-invariant *of f, and* $\deg\lambda$ *the* λ-invariant *of f.*

In particular, Λ is a UFD. Its height 1 prime ideals are simply (p) and $(f(T))$ for f irreducible distinguished polynomials. Hence all the localizations $\Lambda_{\mathfrak{p}}$ at height 1 prime ideals are DVRs.[4]

Definition 2.2. A Λ-module M is *pseudo-null*[5] if it is annihilated by some power of $\mathfrak{m}$. A *pseudo-isomorphism* is a morphism $M_1\to M_2$ with pseudo-null kernel and cokernel.

Remark 2.3. If there is a pseudo-isomorphism $M_1\to M_2$, it is not true that there must be a pseudo-isomorphism $M_2\to M_1$. But this *is* true if M_1 and M_2 are finitely generated torsion Λ-modules, where pseudo-isomorphism gives an equivalence relation.

[3] [Was97, Section 13.2] or [Sha, Section 2.4] contain proofs for the statements in this section.
[4] More generally, $\mathbb{Z}_p[\![T_1,\dots,T_n]\!]$ is still a *Krull domain*, a certain higher dimension generalization of Dedekind domains
[5] A module over a Krull domain is said to be pseudo-null if its annihilator ideal has height ≥ 2.

We note that a Λ-module M has finite cardinality if and only if it is finitely generated and pseudo-null. We have the following analogue of the structure theorem for finitely generated modules over PIDs.[6]

Theorem 2.4. *Let M be a finitely generated Λ-module. Then there is a pseudo-isomorphism*

$$M \to \Lambda^r \oplus \bigoplus_i \Lambda/f_i^{e_i}\Lambda$$

for some $r \geq 0$ and f_i are finitely many irreducible elements. r is determined by M and is additive on exact sequences. If $r = 0$, then f_i and e_i are uniquely determined.

We define

Definition 2.5. For M a finitely generated torsion Λ-module, we define its *characteristic ideal* $\mathrm{Ch}(M) = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{length}_{\Lambda_{\mathfrak{p}}} M\otimes_\Lambda \Lambda_{\mathfrak{p}}}$

By definition, the characteristic ideal is multiplicative in exact sequences of finitely generated torsion Λ-modules. Moreover, for M finitely generated torsion, M is pseudo-null exactly if $\mathrm{Ch}(M) = \Lambda$. Thus

Proposition 2.6. *If $M \to \bigoplus_i \Lambda/f_i^{e_i}$ as above is a pseudo isomorphism, then* $\mathrm{Ch}(M) = \left(\prod_i f_i^{e_i}\right)$.

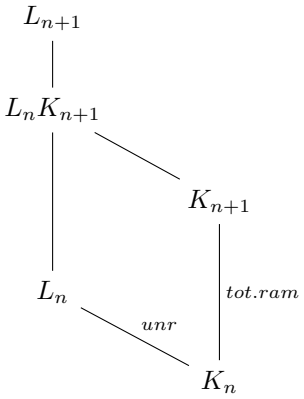
3. The descent procedure

Let's now come back to the case that $X_n = \mathrm{Cl}(K_n)[p^\infty]$ for $K_n = \mathbb{Q}(\mu_{p^{n+1}})$. We formed $X_\infty = \varprojlim_n X_n$ under norms. How can we hope to recover X_n? By the definition of X_∞, we have a natural map

$$X_\infty \to X_n.$$

Proposition 3.1. *The natural map $X_\infty \to X_n$ is surjective.*

Proof. In fact, we will prove that $\mathrm{Nm}_{K_{n+1}/K_n} \colon X_{n+1} \to X_n$ is surjective for all $n \geq 0$. This will rely on the fact that p is totally ramified in K_{n+1}.[7] Let L_n denote the maximal unramified abelian p-extension of K_n. Then we have the diagram, where labels denote the behaviour of primes above p.



[6]This also holds over Krull domains, although it is not true that pseudo-null is the same as finite cardinality.
[7]This is not true for all $\mathbb{Z}_p$ extensions. For instance, it is not true for K_∞^{anti}/K for a quadratic imaginary field K.

By ramification reasons, we must have $L_n \cap K_{n+1} = K_n$. Thus

$$X_{n+1} = \mathrm{Gal}(L_{n+1}/K_{n+1}) \twoheadrightarrow \mathrm{Gal}(L_nK_{n+1}/K_{n+1}) = \mathrm{Gal}(L_n/K_n) = X_n$$

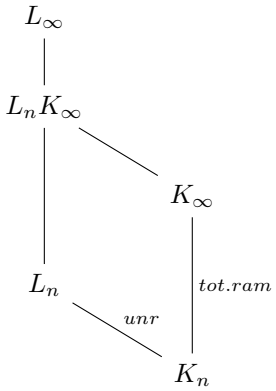
and such map is identified with $\mathrm{Nm}_{K_{n+1}/K_n} \colon X_{n+1} \to X_n$. □

Proposition 3.2 ([Was97, Proposition 13.22])**.** *We have $X_n = X_\infty/\nu_n X_\infty$ where*

$$\nu_n := (1+T)^{p^n} - 1 \in \Lambda.$$

Proof. Recall that $1 + T = \gamma$, and thus $\alpha := 1 + \nu_n$ is a topological generator of $\mathrm{Gal}(K_\infty/K_n)$.

Consider the diagram as in the previous proof



Then $G := \mathrm{Gal}(L_\infty/K_n) = X_\infty \hat{\rtimes} \langle \alpha \rangle$ for a choice of lift of α. L_n is the maximal unramified abelian subextension of L_∞/K_n, so

$$X_n = \mathrm{Gal}(L_n/K_n) = (X_\infty \hat{\rtimes} \langle \alpha \rangle)/\overline{([G,G],\alpha)} = X_\infty/(g \sim \alpha \cdot g \colon g \in X_\infty) = X_\infty/\nu_n X_\infty,$$

as $\alpha^{-1} g \alpha g^{-1} \in [G,G]$ and thus we must have $\alpha \cdot g = \alpha^{-1} g \alpha \sim g$. □

Corollary 3.3. *X_∞ is a finite generated torsion Λ-module.*

Proof. As $X_0/pX_0 = X_\infty/\mathfrak{m}X_\infty$ is finite, we conclude that X_∞ is a finitely generated Λ-module by Nakayama. It is also Λ-torsion as X_0 is finite. □

Now given $\chi = \omega^i$ a power of the Teichmüller character, assume that we had a pseudo-isomorphism $X_\infty^\chi \to \bigoplus \Lambda/f_i$. Then we can consider the diagram

$$\begin{array}{ccc} X_\infty^\chi & \longrightarrow & \bigoplus \Lambda/f_i \\ \downarrow{\scriptstyle \cdot \nu_n} & & \downarrow \\ X_\infty^\chi & \longrightarrow & \bigoplus \Lambda/f_i \end{array}$$

to try to compare $X_n^\chi = X_\infty^\chi/\nu_n X_\infty^\chi$ and $\bigoplus \Lambda/(f_i, \nu_n)$. Following this, one can prove

Lemma 3.4 ([Was97, Theorem 13.13])**.** *If X is a finitely generated torsion Λ-module with $X/\nu_n X$ finite for all $n \geq 0$, then there is $n_0 \geq 0$ and $c \in \mathbb{Z}$ such that*

$$\#X/\nu_n X = p^{np^\mu + n\lambda + c} \text{ for all } n \geq n_0,$$

where μ, λ are the invariants of $\mathrm{Ch}(X)$.

But often we can be more precise than that. The main issue for the ambiguity in the lemma above is that $X \to \bigoplus \Lambda/f_i$ in general can have both a kernel and cokernel. But fortunately, often for the modules in Iwasawa theory the kernel must be 0. For example:

Proposition 3.5. *X_∞^χ has no nonzero pseudo-null submodules.*

Proof. If it did contain a nonzero pseudo-null submodule Y, then $\mathfrak{m}^k Y = 0$ for some k. So it suffices to prove that if $Y \subseteq X_\infty^\chi$ is a submodule with $\mathfrak{m}Y = 0$, then $Y = 0$. If $c = (c_n)_{n\geq 0} \in Y$, then $pc = 0$, and thus $c_n \in \mathrm{Cl}(K_n)[p]$ for all n. As $Tc = 0$, we also have $(\gamma - 1)c = 0$ for any $\gamma \in \mathrm{Gal}(K_\infty/K_0)$. So $c_n \in \mathrm{Cl}(K_n)[p]^{G_{K_0}}$ But then $c_n = \mathrm{Nm}_{K_{n+1}/K_n} c_{n+1} = p \cdot c_{n+1} = 0$ for all $n \geq 0$. □

Corollary 3.6. *We have $\#X_n^\chi = \prod_i \#\Lambda/(f_i, \nu_n)$. In particular, $\#X_0^\chi = \#\mathbb{Z}_p/\mathrm{Ch}(X_\infty^\chi)(0)$.*

Proof. This follows from applying the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_\infty^\chi & \longrightarrow & \bigoplus_i \Lambda/f_i & \longrightarrow & \mathrm{coker} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot\nu_n} & & \downarrow{\scriptstyle \cdot\nu_n} & & \downarrow{\scriptstyle \cdot\nu_n} & & \\ 0 & \longrightarrow & X_\infty^\chi & \longrightarrow & \bigoplus_i \Lambda/f_i & \longrightarrow & \mathrm{coker} & \longrightarrow & 0 \end{array}$$

Since X_n^χ is finite, the Snake lemma implies that $\Lambda/(f_i, \nu_n)$ must have finite cardinality. This means that f_i and ν_n are coprime, and hence that $\ker(\Lambda/f_i \xrightarrow{\cdot\nu_n} \Lambda/f_i) = 0$. Now the claim follows from the Snake lemma by noting that $\mathrm{coker}[\nu_n]$ and coker/ν_n have the same cardinality as coker has finite cardinality. □

Recall that we should have

$$\mathrm{Cl}(\mathbb{Q}(\mu_p))[p^\infty]^\chi = \begin{cases} 0 & \text{if } \chi = \omega, \\ |L(0, \chi^{-1})|_p & \text{if } \chi \text{ is odd and } \chi \neq \omega, \\ |(\mathcal{O}^\times_{\mathbb{Q}(\mu_p)^+}/C)^\chi|_p & \text{if } \chi \text{ is even.} \end{cases}$$

We proved this for χ even using Euler systems, but historically it was first deduced from Mazur–Wiles proof of:

Conjecture 3.7 (Iwasawa Main Conjecture)**.** *Let E_n denote the units of K_n^+ that are congruent to 1 modulo the prime above p. Let $C_n \subseteq E_n$ be the subset of cyclotomic units. Denote E_∞, C_∞ their limits under the norm map. For χ even nontrivial, denote also $\mathscr{L}_{KL}^\chi \in \Lambda$ the Kubota–Leopoldt p-adic L function for χ. Then for $\chi \neq \omega^0, \omega^1$, we have*

$$\mathrm{Ch}(X_\infty^\chi) = \begin{cases} (\mathscr{L}_{KL}^{\omega\chi^{-1}}) & \text{if } \chi \text{ is odd,} \\ \mathrm{Ch}(E_\infty/C_\infty)^\chi & \text{if } \chi \text{ is even.} \end{cases}$$

Here, for χ even nontrivial, the Kubota–Leopoldt p-adic L-function is the unique element $\mathscr{L}_{KL}^{\chi} \in \Lambda$ such that $\epsilon_{cycl}^{n}(\mathscr{L}_{KL}^{\chi}) = L^*(n, \chi\omega^{n-1})$ for all $n \leq 0$. For an explicit construction of element, see [Was97, Theorem 7.10]. We will later give another way to construct this.

In fact, the Euler system argument we gave can be adapted to prove the above conjecture when χ is even: see [Was97, Section 15] for details. We will explain how, in fact, the two parts of the main conjecture are *equivalent*. This is often called the *reflection theorem* in this classical context. We will see next week how this is a particular case of a more general philosophy connecting Euler systems and Iwasawa main conjectures.

To build up for the proof of the reflection theorem, we will reinterpret the modules we have been considering in terms of Selmer groups.

4. In terms of Selmer groups

Suppose we have a p-adic representation V with a G_K-stable lattice Λ. Denote $W := V/\Lambda$. From the exact sequence $0 \to \Lambda \to V \to W \to 0$, we have for a place v

$$H^1(K_v, \Lambda) \xrightarrow{\alpha} H^1(K_v, V) \xrightarrow{\beta} H^1(K_v, W).$$

A Selmer structure on $H^1_{\mathcal{L}}(K_v, V)$ can be *propagated* to $H^1(K_v, \Lambda)$ and $H^1(K_v, W)$ simply by defining

$$H^1_{\mathcal{L}}(K_v, \Lambda) := \alpha^{-1}(H^1_{\mathcal{L}}(K_v, V)), \quad H^1_{\mathcal{L}}(K_v, W) := \beta(H^1_{\mathcal{L}}(K_v, V)).$$

We will look mostly at $H^1_{\mathcal{L}}(K, W)$. Recall from Gefei's talk

Proposition 4.1. *The Kummer map induces an isomorphism $\mathcal{O}_K^{\times} \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1_f(K, \mathbb{Q}_p(1))$. For an elliptic curve E/K, the Kummer map $E(K) \otimes \mathbb{Q}_p \to H^1_f(K, V_pE)$ is an isomorphism if and only if $Ш(E/K)[p^\infty]$ is finite.*

But in fact, we actually have

Proposition 4.2. *The inverse limit of the finite level Kummer maps identify $\mathcal{O}_K^{\times} \otimes \mathbb{Z}_p \xrightarrow{\sim} H^1_f(K, \mathbb{Z}_p(1))$. The direct limit of the finite level Kummer map fits into an exact sequence*

$$0 \to \mathcal{O}_K^{\times} \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1_f(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \to \mathrm{Cl}(K)[p^\infty] \to 0.$$

Similarly, if E is an elliptic curve over K, then the natural map $E(K) \otimes \mathbb{Z}_p \hookrightarrow H^1_f(K, T_pE)$ is an isomorphism iff $Ш(E/K)[p^\infty]$ is finite, and we also have that $H^1_f(K, E[p^\infty]) = \mathrm{Sel}_{p^\infty}(E/K)$ fits into the exact sequence

$$0 \to E(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1_f(K, E[p^\infty]) \to Ш(E/K)[p^\infty] \to 0.$$

Let's also look at the trivial representation $\mathbb{Q}_p$. Since its weight is 0, the Bloch–Kato conditions are unramified everywhere. The propagations to $\mathbb{Z}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p$ can be checked to also be just the unramified cohomology. Thus

$$H^1_f(K, \mathbb{Q}_p) = H^1_f(K, \mathbb{Z}_p) = 0, \quad H^1_f(K, \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}(\mathrm{Cl}(K), \mathbb{Q}_p/\mathbb{Z}_p).$$

So X_∞ is identified with

$$\mathrm{Hom}\left(\varinjlim_n H^1_f(K_n, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p\right),$$

where the transition maps are simply the restriction.

Following Greenberg, we an give a different description of this direct limit.

Proposition 4.3. *Let V be a p-adic representation of G_K unramified away from Σ with G_K-stable lattice T. Denote $W = V/T$. Let K_∞/K be an abelian tower of finite extensions K_n/K unramified away from Σ. Let $\Lambda_{K_\infty/K} := \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/K)]\!]$, and $\Lambda^\vee_{K_\infty/K} := \mathrm{Hom}(\Lambda_{K_\infty/K}, \mathbb{Q}_p/\mathbb{Z}_p)$ as G_K-modules, and Λ^{cycl}-action by $(\lambda \cdot f)(x) = f(x\lambda)$. Let $\mathbb{T}_T := T \otimes_{\mathbb{Z}_p} \Lambda_{K_\infty/K}$ and $\mathbb{W}_T := T \otimes_{\mathbb{Z}_p} \Lambda^\vee_{K_\infty/K}$. Then*

$$\varprojlim_n H^1(K_\Sigma/K_n, T) = H^1(K_\Sigma/K, \mathbb{T}_T) \quad \text{and} \quad \varinjlim_n H^1(K_\Sigma/K_n, W) = H^1(K_\Sigma/K, \mathbb{W}_T).$$

Proof. We only prove the second equality, since the first is analogous.

By Shapiro's lemma, we have $H^1(K_\Sigma/K_n, W) = H^1(K_\Sigma/K, \mathrm{Ind}_{G_{K_n}}^{G_K} W)$. So It suffices to see that $\varinjlim_n \mathrm{Ind}_{G_{K_n}}^{G_K} W = \mathbb{W}$ as G_K-modules. We have

$$\mathrm{Ind}_{G_{K_n}}^{G_K} W = \{f\colon G_K \to W\colon f(\sigma x) = f(x)^\sigma \text{ for } x \in G_K,\ \sigma \in G_{K_n}\}$$

and so

$$\varinjlim_n \mathrm{Ind}_{G_{K_n}}^{G_K} W = \mathrm{Hom}(\Lambda_{K_\infty/K}, W)$$

which is $\mathbb{W}_T$ as $W = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. □

One can define Selmer structures on these cohomology groups by the inverse/direct limit of the Bloch–Kato local conditions.[8] Then we indeed have $H^1_f(K, \mathbb{T}_T) = \varprojlim H^1_f(K_n, T)$ and $H^1_f(K, \mathbb{W}_T) = \varinjlim H^1_f(K_n, W)$.

Definition 4.4. We denote $\mathrm{Sel}(T) = H^1_f(\mathbb{Q}, \mathbb{T}_T)$, $S(T) = H^1_f(\mathbb{Q}, \mathbb{W}_T)$ and $X(T) = \mathrm{Hom}(S(T), \mathbb{Q}_p/\mathbb{Z}_p)$ when the extension K_∞/K is implied.

Example 4.5. For $T = \mathbb{Z}_p$ and $T = \mathbb{Z}_p(1)$, we have

$$\mathrm{Sel}(\mathbb{Z}_p) = 0, \quad \mathrm{Sel}(\mathbb{Z}_p(1)) = \varprojlim_n (\mathcal{O}^\times_{K_n} \otimes \mathbb{Z}_p), \quad X(\mathbb{Z}_p) = \varprojlim_n \mathrm{Cl}(K_n)[p^\infty],$$

and $X(\mathbb{Z}_p(1))$ fits in the exact sequence

$$0 \to \left(\varinjlim_n \mathrm{Cl}(K_n)[p^\infty]\right)^\vee \to X(\mathbb{Z}_p(1)) \to \left(\varinjlim_n (\mathcal{O}^\times_{K_n} \otimes \mathbb{Z}_p)\right)^\vee \to 0$$

5. Reflection theorem

Let's return to the case $K_n = \mathbb{Q}(\mu_{p^n})$.

[8] To be precise, one needs to consider the inverse/direct limit of the semi-local cohomology groups: for a place v of K, consider $H^1_f(K_{n,v}, ?) := \bigoplus_{w|v \text{ in } K_n} H^1_f(K_{n,w}, ?)$.

5.1. **Local conditions.** We think of Λ^{cycl} as a *p-adic interpolation of the Tate twists* $\mathbb{Z}_p(k)$. Indeed, we have $G_{\mathbb{Q}}$-equivariant specializations $\mathrm{sp}_k\colon \Lambda^{cycl} \to \mathbb{Z}_p(k)$ given by $g \mapsto \epsilon_{cycl}^k(g)$. So we note the following quite confusing fact:

Proposition 5.1. *$H^1_{f,\{p\}}(\mathbb{Q}, \mathbb{T}_{\mathbb{Z}_p(1)}) = H^1_{f,\{p\}}(\mathbb{Q}, \mathbb{T}_{\mathbb{Z}_p}) \otimes \epsilon_{cycl}^{-1}$ as Λ^{cycl}-modules. Similarly, $H^1_{f,\{p\}}(\mathbb{Q}, \mathbb{W}_{\mathbb{Z}_p(1)}) = H^1_{f,\{p\}}(\mathbb{Q}, \mathbb{W}_{\mathbb{Z}_p}) \otimes \epsilon_{cycl}$ as Λ^{cycl}-modules.*

Proof. We have $\mathbb{T}_{\mathbb{Z}_p(1)} = \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \Lambda^{cycl} = \mathbb{Z}_p \otimes_{\mathbb{Z}_p} \Lambda^{cycl}(1)$. But note that we have a $G_{\mathbb{Q}}$-equivariant isomorphism of Λ^{cycl}-modules $\Lambda^{cycl}(1) \xrightarrow{\sim} \Lambda^{cycl}(\epsilon_{cycl}^{-1})$ where ϵ_{cycl} denotes a twist only on the Λ^{cycl}-action, not on the $G_{\mathbb{Q}}$ action. This is simply given by $g \mapsto \epsilon_{cycl}^{-1}(g)g$. Hence $\mathbb{T}_{\mathbb{Z}_p(1)} \xrightarrow{\sim} \mathbb{T}_{\mathbb{Z}_p} \otimes \epsilon_{cycl}^{-1}$. Similarly, $\mathbb{W}_{\mathbb{Z}_p(1)} \xrightarrow{\sim} \mathbb{W}_{\mathbb{Z}_p} \otimes \epsilon_{cycl}$. Finally, one can check that the local conditions outside p agree, since they are in fact trivial for both $\mathbb{W}_{\mathbb{Z}_p}$ and $\mathbb{W}_{\mathbb{Z}_p(1)}$, as we explain in what follows.

In fact, if $l \neq p$, then $H^1_f(\mathbb{Q}_l, \mathbb{W}_T) = 0$ for any T. If p^e is the largest power of p that divides $l-1$, then l splits completely over $K_e/\mathbb{Q}$, and each prime λ above l is totally inert in K_∞/K_e. Fix such λ, and let λ_n be the unique prime above it in K_n. We are looking at $\varinjlim_n H^1(k(\lambda_n), W^{I_{\lambda_n}})$. Now for any $c_n \in H^1(k(\lambda_n), W^{I_{\lambda_n}})$, choose a large enough so that $c_n(\mathrm{Frob}_{\lambda_n})$ is fixed by $G_{K_{n+a}}$. Then $c_n(\mathrm{Frob}_{\lambda_{n+a}}) = \mathrm{Nm}_{K_{n+a}/K_n} c_n(\mathrm{Frob}_{\lambda_n})$ by the cocycle condition. Choose b such that $p^b c_n(\mathrm{Frob}_{\lambda_n}) = 0$. Then the above says that the restriction of c_n to $H^1(k(\lambda_{n+a+b}), W^{I_{\lambda_{n+a+b}}})$ is zero. □

Now let's discuss the local conditions above p.

Proposition 5.2. *$H^1_f(\mathbb{Q}_p, \mathbb{T}_{\mathbb{Z}_p}) = 0$ and $H^1_f(\mathbb{Q}_p, \mathbb{W}_{\mathbb{Z}_p(1)}) = H^1(\mathbb{Q}_p, \mathbb{W}_{\mathbb{Z}_p(1)})$.*

Proof. We have

$$H^1_f(K_{n,p}, \mathbb{Z}_p) = H^1_{unr}(K_{n,p}, \mathbb{Z}_p) = H^1(\mathbb{F}_{(p-1)p^n}, \mathbb{Z}_p) = \mathrm{Hom}(G_{\mathbb{F}_{(p-1)p^n}}, \mathbb{Z}_p)$$

but then the transition maps are identified with the restrictions $G_{\mathbb{F}_{(p-1)p^n}} \to G_{\mathbb{F}_{(p-1)p^{n+1}}}$. And then we conclude $H^1_f(\mathbb{Q}_p, \mathbb{T}_{\mathbb{Z}_p}) = \mathrm{Hom}(G_{\mathbb{F}_{(p-1)p^\infty}}, \mathbb{Z}_p) = 0$.

The second claim follows from local duality. □

For $\mathbb{Z}_p(1)$, the local condition at p is more subtle: we have $0 \to \mu_{p-1} \to \mathcal{O}^\times_{\mathbb{Q}_p(\mu_{p^n})} \to H^1_f(\mathbb{Q}_p(\mu_{p^n}), \mathbb{Z}_p(1)) \to 0$, and so we are looking at $\varprojlim_{\mathrm{Nm}}(\mathcal{O}^\times_{\mathbb{Q}_p(\mu_{p^n})})$. This module can be very concretely described, as done by Coleman:

Theorem 5.3 ([Sha, Theorem 5.4.31])**.** *Fix a choice of norm-compatible roots of unity ζ_{p^n}. Then there exist an exact sequence of Λ^{cycl}-modules*

$$0 \to \mu_{p-1} \times \mathbb{Z}_p(1) \xrightarrow{(\xi,a)\mapsto(\xi\zeta_{p^n}^a)_n} \varprojlim_{\mathrm{Nm}}(\mathcal{O}^\times_{\mathbb{Q}_p(\mu_{p^n})}) \xrightarrow{\mathrm{Col}} \Lambda^{cycl} \xrightarrow{\epsilon_{cycl}} \mathbb{Z}_p(1) \to 0.$$

The map Col is explicit, and we have explicit norm compatible cyclotomic units $C_\infty \subseteq \varprojlim_{\mathrm{Nm}}(\mathcal{O}^\times_{\mathbb{Q}_p(\mu_{p^n})})$. One can compute their image on the Coleman map:

Theorem 5.4 (Explicit reciprocity law, [Sha, Theorem 6.13]). *If $\chi\colon \Delta \to \mathbb{Z}_p^\times$ is even and nontrivial, then the image of $\mathrm{Col}(C_\infty^\chi) \in \Lambda^{cycl,\chi} = \Lambda$ is generated by a function $f(T)$ with $f((1+p)^k - 1) = L^*(1-k, \chi\omega^{-k})$ for all $k > 0$. In particular, we must have $\epsilon_{cycl}^k(f) = \epsilon_{cycl}^{1-k}(\mathscr{L}_{KL}^\chi)$ for all $k \in \mathbb{Z}$.*

This result is a very explicit computation. It is also constructing the Kubota–Leopoldt p-adic L-function! Moreover, it gives an interpretation of $\epsilon_{cycl}^k(\mathscr{L}_{KL}^\chi)$ for $k \in \mathbb{Z}$ *outside* the range of interpolation. For instance, it recovers the following formula.

Corollary 5.5 (Leopoldt). *For $\chi\colon \Delta \to \mathbb{Z}_p^\times$ a nontrivial even character,*

$$\epsilon_{cycl}(\mathscr{L}_{KL}^\chi) = \frac{\sum_{a=1}^{p-1} \chi^{-1}(a) \log_p(1-\zeta_p^a)}{\sum_{a=1}^{p-1} \chi^{-1}(a)\zeta_p^a}.$$

5.2. **Reflection theorem.** By the analysis of the local conditions above, we have

$$0 \to \mathrm{Sel}(\mathbb{Z}_p) \otimes \epsilon_{cycl}^{-1} \to \mathrm{Sel}(\mathbb{Z}_p(1)) \xrightarrow{\mathrm{loc}_p} H_f^1(\mathbb{Q}_p, \mathbb{T}_{\mathbb{Z}_p(1)})$$

and

$$0 \to S(\mathbb{Z}_p) \otimes \epsilon_{cycl} \to S(\mathbb{Z}_p(1)) \xrightarrow{\mathrm{loc}_p} H_{/f}^1(\mathbb{Q}_p, \mathbb{W}_{\mathbb{Z}_p}) \otimes \epsilon_{cycl}.$$

We can piece these together by global duality. Since $\mathrm{Sel}(\mathbb{Z}_p) = 0$, we get

$$0 \to \mathrm{Sel}(\mathbb{Z}_p(1)) \xrightarrow{\mathrm{loc}_p} H_f^1(\mathbb{Q}_p, \mathbb{T}_{\mathbb{Z}_p(1)}) \xrightarrow{\mathrm{loc}_p^\vee} X(\mathbb{Z}_p(1)) \otimes \epsilon_{cycl} \to X(\mathbb{Z}_p) \to 0.$$

Dividing by the cyclotomic units, we get

$$0 \to \frac{\mathrm{Sel}(\mathbb{Z}_p(1))}{C_\infty} \xrightarrow{\mathrm{loc}_p} \frac{H_f^1(\mathbb{Q}_p, \mathbb{T}_{\mathbb{Z}_p(1)})}{\mathrm{loc}_p(C_\infty)} \xrightarrow{\mathrm{loc}_p^\vee} X(\mathbb{Z}_p(1)) \otimes \epsilon_{cycl} \to X(\mathbb{Z}_p) \to 0.$$

Since C_∞^χ is only nonzero if $\chi\colon \Delta \to \mathbb{Z}_p^\times$ is even and nontrivial, let's take such χ and consider

$$0 \to \frac{\mathrm{Sel}(\mathbb{Z}_p(1))^\chi}{C_\infty^\chi} \xrightarrow{\mathrm{loc}_p} \frac{H_f^1(\mathbb{Q}_p, \mathbb{T}_{\mathbb{Z}_p(1)})^\chi}{\mathrm{loc}_p(C_\infty^\chi)} \xrightarrow{\mathrm{loc}_p^\vee} X(\mathbb{Z}_p(1))^{\chi\omega^{-1}} \otimes \epsilon_{cycl} \to X(\mathbb{Z}_p)^\chi \to 0.$$

Now the explicit reciprocity law says that the second Λ-module is torsion. We already known the last one is also torsion. So all four modules are torsion, and we can compare their characteristic ideals.

From the description of $X(\mathbb{Z}_p(1))$, note that since $\chi\omega^{-1}$ is odd and not ω^{-1}, we have

$$\mathrm{Hom}\left(\varinjlim_n \mathrm{Cl}(K_n)[p^\infty]^{\omega\chi^{-1}}, \mathbb{Q}_p/\mathbb{Z}_p\right) \xrightarrow{\sim} X(\mathbb{Z}_p(1))^{\chi\omega^{-1}}$$

An exercise in algebra let us conclude from this that $\mathrm{Ch}\left(X(\mathbb{Z}_p(1))^\chi\right) = \iota(\mathrm{Ch}(X_\infty^{\chi^{-1}}))$, where $\iota\colon \Lambda \to \Lambda$ is the involution given by inversion $\iota(g) = g^{-1}$. More generally, the following is true.

Proposition 5.6 ([Was97, Proposition 15.32]). *If X is a finitely generated torsion Λ-module with $X/\nu_n X$ finite, then* $\mathrm{Ch}\left(\mathrm{Hom}(\varinjlim X/\nu_n X, \mathbb{Q}_p/\mathbb{Z}_p)\right) = \iota(\mathrm{Ch}(X))$.

The explicit reciprocity law says that

$$\mathrm{Ch}\left(\frac{H^1_f(\mathbb{Q}_p, \mathbb{T}_{\mathbb{Z}_p(1)})^\chi}{\mathrm{loc}_p(C^\chi_\infty)}\right) = (\mathrm{Tw} \circ \iota)(\mathscr{L}^\chi_{KL})$$

where $\mathrm{Tw}\colon \Lambda \to \Lambda$ is $g \mapsto \epsilon_{cycl}(g)g$. So the above exact sequence tells us that

$$\frac{\mathrm{Ch}(E_\infty/C_\infty)^\chi}{\mathrm{Ch}(X^\chi_\infty)} = (\mathrm{Tw} \circ \iota)\left(\frac{(\mathscr{L}^\chi_{KL})}{\mathrm{Ch}(X^{\omega\chi^{-1}}_\infty)}\right).$$

That is, this proves:

Theorem 5.7 (Reflection Theorem)**.** *For $\chi \neq \omega^0, \omega^1$, the Iwasawa main conjecture for χ and $\omega\chi^{-1}$ are equivalent.*

November 28: Iwasawa theory of elliptic curves

Warning: These notes are meant to give a big picture overview of the subject. I will not try to spell out all the technical assumptions for the "big" theorems in this exposition, and many claims will only be approximately correct. For precise result, one should follow the references given.

6. General philosophy

Let $T \subseteq V$ be a lattice inside a geometric p-adic representation, and denote $W = V/T$. We consider the Bloch–Kato Selmer groups $H^1_f(F, ?)$ for $? \in \{T, V, W\}$. The group $H^1_f(F, W)$ contains interesting information besides just the dimension $\dim H^1_f(F, V)$. Namely, we define

$$Ш_f(W/F) := H^1_f(F, W)_{/\mathrm{div}}$$

where the subscript means the quotient by the maximal divisible submodule (which is the image of $H^1_f(F, V)$).

Example 6.1. If $T = \mathbb{Z}_p(1)$, then $Ш_f(W/F) = \mathrm{Cl}(F)[p^\infty]$. If $T = \mathbb{Z}_p$, then $Ш_f(W/F) = \mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Q}_p/\mathbb{Z}_p)$. If $T = T_pE$, then $Ш_f(W_pE/F) = Ш(E/F)[p^\infty]_{/\mathrm{div}}$, which is, of course, $Ш(E/F)[p^\infty]$ if this is finite.

Paraphrasing Kato, there are three phases of understanding of special values of L-functions. Here we think of V to be the p-adic realization of some motive.

(0) The Bloch–Kato conjecture predicts the order of vanishing $\mathrm{ord}_{s=0}L(s, V)$ to be $\dim H^1_f(F, V^*(1)) - \dim H^0(F, V^*(1))$. So let's assume this is 0.
(1) $L(0, V)$ is often algebraic except for certain *periods*. In some cases, Deligne and Beillison conjecture certain periods $\Omega_{V,r}$, such that $L(0, V) \in \Omega_{V,r} \cdot \overline{\mathbb{Q}}^\times$. We will denote $L(0, V)/\Omega_{V,r}$ by $L(0, V)_{alg}$.
(2) As we vary V in some suitable p-adic family, the values $L(0, V)_{alg}$ often vary p-adically as well.
(3) The value $L(0, V)_{alg}$ often have deep arithmetic significance.

Example 6.2. Last week we saw this for the family $\mathbb{Q}_p(k) \otimes \omega^k\chi$ for $\chi\colon \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \to \mathbb{Z}_p^\times$ as we vary $k \leq 0$. We have $L(0, \mathbb{Q}_p(k) \otimes \omega^k\chi) = L(k, \omega^k\chi)$, which is nonzero only if χ is odd. Then these values are exactly what are interpolated by $\mathscr{L}_{KL}^{\chi\omega^{-1}}$. As we discussed, this p-adic L-function is deeply related to the p-primary part of class groups of p-power cyclotomic fields.

This is still quite vague, so let's start to get more concrete. Let K_∞/K be a $\mathbb{Z}_p^d$-extension and denote $\Gamma := \mathrm{Gal}(K_\infty/K)$. Let $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$ be the Iwasawa algebra. It is isomorphic to $\mathbb{Z}_p[\![T_1, \ldots, T_d]\!]$.

For a suitable subset $\Xi \subseteq \mathrm{Hom}_{cont}(\Gamma, \overline{\mathbb{Q}_p}^\times)$ of characters, we will consider the p-adic representations $V(\chi)$ as $\chi \in \Xi$. Here $V(\chi)$ means twisting V by $G_K \twoheadrightarrow \Gamma \xrightarrow{\chi} \overline{\mathbb{Q}}_p^\times$ (after extending scalars to contain the image of χ). From the specialization morphisms $\chi\colon \Lambda \to \overline{\mathbb{Q}_p}^\times$, we have maps $H^1(K, \mathbb{T}_T) \to H^1(K, T(\chi))$, and we can define a Selmer group $\mathrm{Sel}_\Xi(T)$ to be the set of classes that specialize to $H^1_f(K, T(\chi))$ for all $\chi \in \Xi$. Similarly we can

define $S_\Xi(T) \subseteq H^1(K, \mathbb{W}_T)$ and $X_\Xi(T) = S_\Xi(T)^\vee$. Of course, this will only actually capture the Selmer groups $H^1_f(K, V(\chi))$ if Ξ is chosen suitably.

Assume that almost all of $L(s, V^*(1)(\chi^{-1}))$ have the same order of vanishing r at $s = 0$. Then we expect $\mathrm{Sel}_\Xi(T)$ and $X_\Xi(T)$ to have Λ-rank r. Furthermore, if $r = 0$, then we can hope that $L(s, V^*(1)(\chi^{-1}))_{alg}$ vary p-adically. That is, that there exist an element $\mathscr{L}_{V,\Xi} \in \Lambda$ such that

$$\chi(\mathscr{L}_{V,\Xi}) = (*) \cdot L(0, V^*(1)(\chi^{-1}))_{alg}$$

up to some simple factors $(*)$. Finally, as we expect that $L(0, V^*(1)(\chi^{-1}))_{alg}$ is related to $Ш_f(W(\chi)/F)$, one can have the hopeful expectation that

$$\mathrm{Ch}(X_\Xi(T)) = (\mathscr{L}_{V,\Xi}).$$

There an ambiguity in this expectation, as the right hand side does not depend on the lattice T. However, different choices of T should only change the left side by a power of p, and we can hope that the choice of T determines a precise choice of period for $\mathscr{L}_{V,\Xi}$.

6.1. **Greenberg Selmer groups.** This is an exposition of the conjectures in [Gre89]. We consider the following condition for a p-adic place v of K.

Definition 6.3. A p-adic representation V of K_v is *ordinary* if there exists a $\mathbb{Q}_p[G_{K_v}]$-stable $\mathbb{Z}$-filtration $F^iV \subseteq V$ that is exhaustive and separated such that the action of inertia in $F^iV/F^{i+1}V$ is by ϵ^i_{cycl}. Denote $V^+ := F^1V$.

In particular, ordinary representations are de Rham with Hodge–Tate weight $-i$ of multiplicity $\dim F^iV/F^{i+1}V$.

Proposition 6.4. *If V is an ordinary K_v-representation, then*

$$H^1_g(K_v, V) = \ker(H^1(K_v, V) \to H^1(I_v, V/V^+)).$$

Proof. First note that $H^1_g(K_v, V/V^+) = H^1_{unr}(K_v, V/V^+)$ by dimension counting, as $(V/V^+)^{G_{K_v}}$ and $D^{\phi=1}_{crys}(V/V^+)$ are 0 because V/V^+ has only strictly positive Hodge–Tate weights. The second assertion follows since $\mathrm{Fil}^1 B^{\phi=1}_{crys} = 0$.

Now in Hao's talk we saw that $H^1(K_v, V \otimes B^+_{dR}) \to H^1(K_v, V \otimes B_{dR})$ is injective for V de Rham, and we also saw that $H^1(K_v, V^+ \otimes B^+_{dR}) = 0$ as V^+ has strictly positive Hodge–Tate weights. Now the claim follows from the commutative diagram

$$\begin{array}{ccccc} & & H^1(K_v, V) & \xrightarrow{\alpha} & H^1(K_v, V/V^+) \\ & & \downarrow & & \downarrow \\ 0 = H^1(K_v, V^+ \otimes B^+_{dR}) & \longrightarrow & H^1(K_v, V \otimes B^+_{dR}) & \hookrightarrow & H^1(K_v, (V/V^+) \otimes B^+_{dR}) \end{array}$$

as then

$$H^1_g(K_v, V) = \alpha^{-1} H^1_g(K_v, V/V^+) = \alpha^{-1} H^1_{unr}(K_v, V/V^+) = \ker\left(H^1(K_v, V) \xrightarrow{\alpha} H^1(K_v, V/V^+) \to H^1(I_v, V/V^+)\right).$$

□

Remark 6.5. For many cases of interest, we have that $H^1_f(K_v, V) = H^1_g(K_v, V)$. By dimension counting, this happens precisely if $D^{\phi=1}_{crys}(V^*(1)) = 0$. For example, this is true if V is pure of weight $w \neq -2$.

For a lattice $T \subseteq V$, we have the induced filtrations $F^iT \subseteq V^iT$, and $F^iW = F^iT \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

Proposition 6.6. *Assume that* $(F^0V/F^1V)^{G_{K_v}} = 0$. *Then* $H^1_g(K_v, W) = \mathrm{im}(H^1(K_v, W^+)_{\mathrm{div}} \to H^1(K_v, W))$. *We also have* $H^1_g(K_v, T) = \ker(H^1(K_v, T) \to H^1(K_v, T/T^+)_{/\mathrm{tor}})$.

Proof. The assumption guarantees that $H^1(K_v, V/V^+) \hookrightarrow H^1(I_v, V/V^+)$. Thus $H^1_g(K_v, V) = \mathrm{im}(H^1(K_v, V^+) \to H^1(K_v, V))$.

For the first claim, consider the commutative diagram

$$\begin{array}{ccc} H^1(K_v, V^+) & \longrightarrow & H^1(K_v, V) \\ \downarrow & & \downarrow \\ H^1(K_v, W^+) & \longrightarrow & H^1(K_v, W) \end{array}$$

So $H^1_g(K_v, W)$ is the image of the above composition. Since the image of the left map is $H^1(K_v, W^+)_{\mathrm{div}}$, the claim follows.

For the second claim, consider the commutative diagram

$$\begin{array}{ccc} H^1(K_v, T) & \longrightarrow & H^1(K_v, V) \\ \downarrow & & \downarrow \\ H^1(K_v, T/T^+) & \longrightarrow & H^1(K_v, V/V^+) \end{array}$$

So $H^1_g(K_v, W)$ is the kernel of the above composition. Since the bottom map has kernel $H^1(K_v, T/T^+)_{\mathrm{tor}}$, the claim follows. □

Take $K = \mathbb{Q}$ and $\mathbb{Q}_\infty$ the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

Then for a $G_{\mathbb{Q}}$-stable lattice $T \subseteq V$, we have the induced filtration F^iT, and Greenberg defines the following Selmer group.[9]

Definition 6.7. $S_{\mathrm{Gr}}(\mathbb{Q}_\infty, W) \subseteq H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathbb{W}_T)$ defined by the local conditions: unramified at $v \nmid p$, and

$$H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathbb{W}_T) := \ker\left(H^1(\mathbb{Q}_p, \mathbb{W}_T) \to H^1(I_p, \mathbb{W}_{T/F^+T})\right).$$

This Selmer group correspond to the subset $\Xi_{\mathrm{Gr}} \subseteq \mathrm{Hom}_{cont}(\Gamma, \mathbb{Z}_p^\times)$ of finite order characters. If we look at $L(s, V(\chi))$ for $\chi \in \Xi_{\mathrm{Gr}}$, then their Archimedean factors are all the same[10], and they have a pole at 0 of order

$$r_V := \sum_{0 \leq k < w/2} m_k(V) + (a^+(V) - m_{w/2}(V)),$$

where we let $a^+(V) = m_{w/2}(V) = 0$ if w is odd. So we expect that $L(s, V(\chi))$ have a zero of order exactly r_V at 0 for all but finitely many χ. So Greenberg conjectures

[9] This is non-standard notation.
[10] As twisting by finite order characters does not change the Hodge–Tate weights

Conjecture 6.8. *For $T \subseteq V$ an ordinary p-adic representation, $X_{\mathrm{Gr}}(\mathbb{Q}_\infty, T)$ is a finitely generated Λ-module of rank $r_{V^*(1)}$.*

The case that $r_V = r_{V^*(1)} = 0$ is exactly the *critical* case considered by Deligne, where the special values are supposed to be algebraic up to a precise period. In the critical case and if V is ordinary, Coates and Perrin-Riou conjecture a precise p-adic interpolation property of $L(0, V(\chi))$. So there is an explicit conjectured p-adic L-function $\mathscr{L}_V \in \mathrm{Frac}(\Lambda)$.

Then Greenberg also conjectures

Conjecture 6.9. *For $T \subseteq V$ an ordinary p-adic representation with $r_V = r_{V^*(1)} = 0$, the characteristic ideal $\mathrm{Ch}(X_{\mathrm{Gr}}(\mathbb{Q}_\infty, T^*(1)))$ is the numerator of $\mathscr{L}_V$ as ideals in $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Here the ambiguity of powers of p come from an ambiguity in the definition of $\mathscr{L}_V$ and also on the choice of lattice T. There should also be a natural way to "normalize" $\mathscr{L}$ with respect to T to get the equality in Λ.

Example 6.10. Take $T = \mathbb{Z}_p(k)$. Then for $\Sigma = \{p, \infty\}$,

$$S_{\mathrm{Gr}}(\mathbb{Q}_\infty, \mathbb{Z}_p(k)) = \begin{cases} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathbb{W}_{\mathbb{Z}_p(k)}) & \text{if } k \geq 1, \\ H^1_{unr}(\mathbb{Q}, \mathbb{W}_{\mathbb{Z}_p(k)}) & \text{if } k \leq 0. \end{cases}$$

So if $X_\infty = \varprojlim_n \mathrm{Cl}(\mathbb{Q}(\mu_{p^n})[p^\infty]$ denotes the $\Lambda[\Delta]$-module of last time, we have $X_{\mathrm{Gr}}(\mathbb{Q}_\infty, \mathbb{Z}_p(k)) = X_\infty^{\omega^k} \otimes \epsilon_{cycl}^k$ if $k \leq 0$, and

$$0 \to \mathrm{Hom}\left((\varinjlim_n \mathrm{Cl}(\mathbb{Q}(\mu_{p^n})[p^\infty])^{\omega^{1-k}},\ \mathbb{Q}_p/\mathbb{Z}_p \right) \otimes \epsilon_{cycl}^{k-1} \to X_{\mathrm{Gr}}(\mathbb{Q}_\infty, \mathbb{Z}_p(k)) \to \mathrm{Hom}\left((\varinjlim_n (\mathcal{O}^{\times, p}_{\mathbb{Q}(\mu_{p^n})})^{\omega^{1-k}},\ \mathbb{Q}_p/\mathbb{Z}_p \right) \otimes \epsilon_{cycl}^{k-1} \to 0$$

if $k \geq 1$. So indeed, we can see that $X_{\mathrm{Gr}}(\mathbb{Q}_\infty, \mathbb{Z}_p(k))$ has rank 1 iff $k \geq 1$ is odd, corresponding to the trivial zero at $1 - k$ for even nontrivial Dirichlet characters. The critical cases are if $k \geq 1$ is even or $k \leq 0$ is odd. So the above conjecture recovers the Iwasawa main conjecture. Note that for $k \leq 0$ even (which is non-critical), the characteristic ideal is not a p-adic L-function.

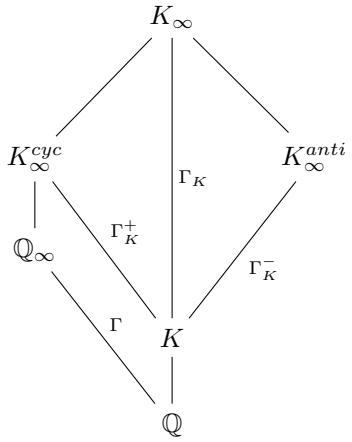
7. Greenberg Selmer groups of elliptic curves

Let $E/\mathbb{Q}$ be an elliptic curve, and $p \geq 5$. We consider $T_pE \subseteq V_pE$. Recall that V_pE is polarized of motivic weight -1, and has Hodge–Tate weights $0, 1$. For simplicity, we will also assume

(irred) $E[p]$ is an irreducible $G_{\mathbb{Q}}$-module.

Let K be a quadratic imaginary field. For simplicity, we will assume that $(N_E, D_K) = 1$, and that $p \nmid D_K$.

We will consider the extensions



and choose topological generators $\gamma, \gamma^+, \gamma^-$ of $\Gamma, \Gamma_K^+, \Gamma_K^-$. So if $T^? = \gamma^? - 1$ for $? \in \{\emptyset, +, -\}$, we have $\Lambda = \mathbb{Z}_p[\![T]\!]$, $\Lambda_K^+ = \mathbb{Z}_p[\![T^+]\!]$, $\Lambda_K^- = \mathbb{Z}_p[\![T^-]\!]$ and $\Lambda_K = \mathbb{Z}_p[\![T^+, T^-]\!]$ the corresponding Iwasawa algebras.

For what follows, let F_∞/F be one of the four extensions above.

7.1. **Greenberg main conjectures.** V_pE is an ordinary representation at a place p exactly when one of the following:

(1) E has good reduction at p and $p \nmid a_p(E)$. That is, E has good non-supersingular reduction.

(2) E has multiplicative reduction.

In the first case, the reduction $\widetilde{E}/\mathbb{F}_p$ has $T_p\widetilde{E} \simeq \mathbb{Z}_p$, and the surjection $T_pE \twoheadrightarrow T_p\widetilde{E}$ give us the filtration. In the second case, the surjection comes from Tate's parametrization $E(\overline{\mathbb{Q}_p}) \simeq \overline{\mathbb{Q}_p}^\times/q^{\mathbb{Z}} \xrightarrow{\text{val}} \mathbb{Q}/e\mathbb{Z}$. There is also a lot of work that has been done to do Iwasawa theory in the case of supersingular reduction, but we will not consider this here for simplicity.[11]

In both cases, V/V^+ is unramified, and Frobenius act by multiplication by α_p. In the first case, α_p is the unit root of $x^2 - a_px + p$, and in the second case it is a_p.

Then we can define the Greenberg local condition at p as before. As for the places not above p, we have:

Proposition 7.1. *Let $v \nmid p$ be a place of F.*

(1) $H^1_{unr}(F_v, \mathbb{W}_T)$ has finite exponent as an abelian group.

(2) If E has good reduction at v, then $H^1_{unr}(F_v, \mathbb{W}_T) = 0$.

(3) If v only has finitely many primes above it in F_∞/F, then $H^1_{unr}(F_v, \mathbb{W}_T) = 0$ as well.

Proof. The third point has the same proof as in the last talk (where we considered the cyclotomic extension of $\mathbb{Q}$).

For L/F_v a finite extension, let $\mathcal{E}$ be the Néron model of E over $\mathcal{O}_L$. Let $\mathcal{E}^0$ be the open subgroup scheme of $\mathcal{E}$ whose generic fiber is E and special fiber is the identity component of the special fiber $\mathcal{E}_0$ of $\mathcal{E}$. Then we have an exact sequence

$$0 \to \mathcal{E}^0(L^{unr}) \to \mathcal{E}(L^{unr}) \to \pi_0(\mathcal{E}_0) \to 0$$

[11] See Skinner's notes [Ski18] for some references.

and $\mathcal{E}(L^{unr}) = E(L^{unr})$. But by Lang's theorem, $H^1(k(L), \mathcal{E}^0(L^{unr})) = 0$. Also, $H^2(k(L), \mathcal{E}^0(L^{unr})) = 0$ because $G_{k(L)}$ has cohomological dimension 1. Thus $H^1_{unr}(L, W) \simeq H^1(k(L), \pi_0(\mathcal{E}_0))[p^\infty]$, which has size $H^0(k(L), \pi_0(\mathcal{E}_0))[p^\infty]$ since $\pi_0(\mathcal{E}_0)$ is finite. In particular, $H^1_{unr}(L, W) = 0$ if E has good reduction over L, and in general $\#\pi_0(\mathcal{E}_0)$ kills $H^1_{unr}(L, W)$ independently of L. Now the first two claims follow from $H^1_{unr}(F_v, \mathbb{W}_T) = \varinjlim_n \bigoplus_{w|v} H^1_{unr}(F_{n,w}, W)$. □

We note that the third condition can only happen if F_∞/F is a $\mathbb{Z}_p$-extension. For $\mathbb{Q}_\infty/\mathbb{Q}$ and K^{cyc}_∞/K, this happens for any $v \nmid p$. For K^{anti}_∞/K, we have the following splitting behaviour for $v \nmid p$: i) if v is split in K, then it is totally inert in K^{anti}_∞/K, ii) if v is inert in K, then it is totally split in K^{anti}_∞/K.

Given this, we may define the following Iwasawa theoretic Selmer groups:

Definition 7.2. Let $S(F_\infty, E) \subseteq H^1(F_\Sigma/F, \mathbb{W}_T)$ be the Selmer group defined by the unramified local conditions for $v \nmid p$, and Greenberg at $v \mid p$. Let also $S^0(F_\infty, E)$ be the Selmer group defined by the trivial local conditions for $v \nmid p$, and Greenberg at $v \mid p$.[12]

Proposition 7.3. *$S^0(F_\infty, E)$ is identified with the direct limit $\varinjlim_{F \subseteq F' \subseteq F_\infty} H^1_f(F', W_p E)$.*

Proof. We have $H^1_f(F_v, W) = 0$ for $v \nmid p$. So it suffices to see that $\varinjlim_n \bigoplus_{w|v} H^1_f(F_{n,w}, W) = H^1_{\mathrm{Gr}}(F_v, \mathbb{W}_T)$. This follows from Shapiro's lemma, Proposition 6.6, the fact that $H^1_?(L, V)$ for $? \in \{e, f, g\}$ are the same for any p-adic field L, as $V \simeq V^*(1)$ are pure of weight -1, and the fact that if $v \mid p$, $\varinjlim_n \bigoplus_{w|v} H^1(F_{n,w}, W^+)_{/\mathrm{div}} = 0$.

For the last claim, note that for a p-adic field L, we have $H^1(L, W^+)_{/\mathrm{div}} \hookrightarrow H^2(L, T^+)_{\mathrm{tor}}$ and $H^2(L, T^+)_{\mathrm{tor}}$ is dual to $H^0(L, W/W^+)_{/\mathrm{div}}$. Thus $\varinjlim_n \bigoplus_{w|v} H^1(F_{n,w}, W/W^+)_{/\mathrm{div}}$ injects into the dual of $\varprojlim_n \bigoplus_{w|v} ((W/W^+)^{G_{F_{n,w}}}_{/\mathrm{div}}$. But for n sufficiently large, all primes of F_n above p are totally ramified along F_∞. So we are looking at $\varprojlim_n ((W/W^+)^{G_{L_n}}_{/\mathrm{div}}$ for a totally ramified $\mathbb{Z}_p^d$-extension L_∞/L of p-adic fields. But W/W^+ is unramified, and the restriction maps $(W/W^+)^{G_{L_n}} \to (W/W^+)^{G_{L_{n+1}}}$ are identities, and thus the inverse limit is 0. □

For $\mathbb{Q}_\infty/\mathbb{Q}$ and K_∞/K, one expects the L-values $L(1, E, \chi) = L(0, V \otimes \chi)$ to be nonzero most of the time for finite order characters. Indeed, we have p-adic L-functions $\mathscr{L}_{\mathbb{Q}_\infty, E} \in \Lambda$ and $\mathscr{L}_{K_\infty, E} \in \Lambda_K$. For example, for a finite order character $\chi \colon \Gamma \to \overline{\mathbb{Q}_p}^\times$ of conductor p^t, we have

$$\chi(\mathscr{L}_{\mathbb{Q}_\infty, E}) = e_p(\chi) \frac{L(1, E, \chi^{-1})}{\Omega_E}, \quad e_p(\chi) = \begin{cases} \alpha_p^{-t} \frac{p^t}{G(\chi^{-1})} & \text{if } t > 0, \\ \left(1 - \frac{1}{\alpha_p}\right)^{2 - \nu_p(N_E)} & \text{if } t = 0. \end{cases}$$

$\mathscr{L}_{\mathbb{Q}_\infty, E}$ was first constructed by Amice–Vélu and Vishik, see [MTT86]. $\mathscr{L}_{K_\infty, E}$ was constructed by Perrin–Riou [PR88].

We have the Iwasawa main Conjectures[13]

Conjecture 7.4 (Cyclotomic main conjecture)**.** *$X(\mathbb{Q}_\infty, E)$ is Λ-torsion, and its characteristic ideal is $(\mathscr{L}_{\mathbb{Q}_\infty, E})$.*

[12] For comparison with [Ski18], $S^0(F_\infty, E)$ corresponds to $S(E/F_\infty)$.
[13] If (irred) does not hold, then the equality of characteristic ideals must be modified by a factor of p

Conjecture 7.5 (Two-variable main conjecture)**.** *$X^0(K_\infty, E)$ is Λ_K-torsion, and its characteristic ideal is $(\mathscr{L}_{K_\infty,E})$.*

Now we restrict to the case of good ordinary reduction. Note that if $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}_\infty$, then by inflation restriction

$$H^1(F/\mathbb{Q}, W^{G_F}) \to H^1(G_{\mathbb{Q},\Sigma}, W) \to H^1(G_{F,\Sigma}, W)$$

and since $W^{G_F} = E(F)[p^\infty]$ is finite and $F/\mathbb{Q}$ is cyclic, $\#H^1(F/\mathbb{Q}, W^{G_F}) = \#\hat{H}^0(F/\mathbb{Q}, W^{G_F})$ and $\hat{H}^0(F/\mathbb{Q}, W^{G_F}) = W^{G_\mathbb{Q}}/\mathrm{Tr}_{F/\mathbb{Q}} W^{G_F} = E(\mathbb{Q})[p^\infty]/\mathrm{Tr}_{F/\mathbb{Q}} E(F)[p^\infty]$. In particular, since we are assuming (irred), we have $H^1_f(\mathbb{Q}, W) \hookrightarrow S(\mathbb{Q}_\infty, E)[T]$. Analyzing it further, one can prove

Proposition 7.6 ([Gre99])**.** *If $X(\mathbb{Q}_\infty, E)$ is Λ-torsion and E has good ordinary reduction at p, then there is an exact sequence*

$$0 \to H^1_f(\mathbb{Q}, W) \to S(\mathbb{Q}_\infty, E)[T] \to \prod_{l \in \Sigma} K_l$$

where $K_l = \ker(H^1_{/f}(\mathbb{Q}_l, W) \to H^1_{/\mathrm{Gr}}(\mathbb{Q}_l, \mathbb{W}_T))$. We have

$$\#K_l = \begin{cases} |c_l(E/\mathbb{Q})|_p^{-1} & \text{if } l \neq p, \\ \#(\mathbb{Z}_p/(1-\alpha_p))^2 & \text{if } l = p. \end{cases}$$

Furthermore, if $H^1_f(\mathbb{Q}, W)$ is finite, then the above exact sequence is also exact on the right.

This implies that

Corollary 7.7. *Assume $X(\mathbb{Q}_\infty, E)$ is Λ-torsion. Then we have*

$$r(E/\mathbb{Q}) = 0 \text{ and } Ш(E/\mathbb{Q})[p^\infty] \text{ is finite} \iff T \nmid \mathrm{Ch}(X(\mathbb{Q}_\infty, E)).$$

Proof. The above implies that

$$H^1_f(\mathbb{Q}, W) \text{ is finite} \iff X^0(\mathbb{Q}_\infty, E)/TX(\mathbb{Q}_\infty, E) \text{ is finite.}$$

But the right hand side is a finite quantity times $\Lambda/(T, \mathrm{Ch}(X(\mathbb{Q}_\infty, E)))$. This is finite if and only if $T \nmid \mathrm{Ch}(X(\mathbb{Q}_\infty, E))$. □

Together with the main conjecture, this would imply the rank 0 case of Bloch–Kato for E:

$$r(E/\mathbb{Q}) = 0 \text{ and } Ш(E/\mathbb{Q})[p^\infty] \text{ is finite} \iff L(1, E) \neq 0.$$

But in the rank 0 case, we can do even better, since

Proposition 7.8 ([Gre99, Proposition 4.8])**.** *$X(\mathbb{Q}_\infty, E)$ has no nonzero pseudo-null submodules.*

So the main conjecture would imply that: if $L(1, E) \neq 0$, then

$$\#\mathbb{Z}_p/\mathrm{triv}(\mathscr{L}_{\mathbb{Q}_\infty,E}) = \#H^1_f(\mathbb{Q}, W) \cdot \prod_{l \in \Sigma} \#K_l,$$

that is, that

$$\#\mathbb{Z}_p/\left(\alpha_p^{-2}(\alpha_p-1)^2\frac{L(1,E)}{\Omega_E}\right) = |Ш(E/\mathbb{Q})|_p^{-1}\cdot\prod_{l|N_E}|c_l(E/\mathbb{Q})|_p^{-1}\cdot\#\mathbb{Z}_p/(1-\alpha_p)^2,$$

which is simply

$$\left|\frac{L(1,E)}{\Omega_E}\right|_p^{-1} = \left|Ш(E/\mathbb{Q})\cdot\prod_{l|N_E}c_l(E/\mathbb{Q})\right|_p^{-1}.$$

This is the p-part of the BSD formula. See [SU14, Theorem 2] for precise results on this.

7.2. **Anticyclotomic extension.** The situation over the anticyclotomic extension is more delicate. Write $N_E = N^+N^-$ where primes in N^+ are split in K and primes in N^- are inert in K. We will assume that N^- is square-free. Then a local root number computation shows that for any $\chi\colon \Gamma_K^- \to \mathbb{Z}_p^\times$ of finite order,

$$\epsilon(E,\chi) = (-1)^{\nu(N^-)+1}.$$

In particular, we can only expect $L(E,\chi,1)$ to be nonzero for almost all χ when N^- is a product of an *odd* number of primes.

7.2.1. **Case $\epsilon = 1$.** In the case $\epsilon = 1$, we have a Jacquet–Langlands transfer of f_E to a definite quaternion algebra B of discriminant $N^-\infty$. Up to a certain normalization, this is a modular form ϕ of level N^+ of B. A formula of Gross, and generalized by Shou-Wu Zhang give us that for $\chi\colon \Gamma_K^- \to \mathbb{Z}_p^\times$ of finite order,

$$\frac{L(E/K,\chi^{-1},1)}{\Omega_E} = \frac{4\eta_{E,N^+,N^-}}{w_K^2\sqrt{-D_K}}|\phi(P_\chi)|^2$$

where P_χ are certain CM cycles, and $\eta_{E,N^+,N^-} \in \mathbb{Z}_p$ is a factor related to the normalization of the Jacquet–Langlands transfer.

The periods $\phi(P_\chi)$ can be p-adically interpolated[14] as in [BD05, Definition 1.6] into a $\mathscr{L}_\phi \in \Lambda_K^-$.

Conjecture 7.9 (Anticyclotomic main conjecture for $\epsilon = 1$)**.** *$X^0(K_\infty^{anti},E)$ is Λ_K^--torsion, and its characteristic ideal is $(\mathscr{L}_\phi)^2$ in $\Lambda_K^-\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.*

Remark 7.10. The factor η_{E,N^+,N^-} is related to the product of Tamagawa factors at primes of N^-, but is not always exactly that, see for example [RT97].

7.2.2. **Case $\epsilon = -1$.** In this case, we expect $X(K_\infty^{anti},E)$ to have rank 1, and we would hope to interpolate $L'(E/K,\chi^{-1},1)$. However, we do not do this directly. For this discussion, we need that N is square-free if $N^- \neq 1$.

A root number computation shows that if $\chi\colon \Gamma_K^- \to \overline{\mathbb{Q}_p}^\times$ is associated to an unramified algebraic Hecke character of infinity type $(n,-n)$ for $n \geq 1$ and $n \equiv 0 \mod (p-1)$. Then the root number of $L(E,\chi^{-1},s)$ is forced to be 1. So consider $\Xi_{BDP} \subseteq \mathrm{Hom}_{cont}(\Gamma_K^-,\overline{\mathbb{Q}_p}^\times)$ the subset of such characters. Then again a Waldspurger-type formula

[14] This requires p to be ordinary.

says that

$$\frac{L(E/K,\chi^{-1},1)}{\Omega_\infty^{4n}} = \eta_{E,N^+,N^-}\cdot(*)\cdot(L(E/K,\chi^{-1},1)_{alg})^2$$

where Ω_∞ is a complex period, and $L(E/K,\chi^{-1},1)_{alg}$ is the result of applying certain powers of the Mass–Shimura operator to the Jacquet–Langlands transfer of f_E, and then evaluate this at a CM divisor determined by χ. Then one can p-adically interpolate, for $\chi\in\Xi_{BDP}$, the quantity

$$e_p(\chi)\frac{L(E/K,\chi^{-1},1)}{\Omega_p^{2n}}$$

where Ω_p is a certain p-adic period and

$$e_p(\chi)=\begin{cases} L(E/K_{\overline{v}},\chi^{-1},1)^{-1} & \text{if } p=v\overline{v} \text{ in } K \text{ where } v \text{ corresponds to } \overline{\mathbb{Q}}\to\overline{\mathbb{Q}_p},\\ 1 & \text{otherwise.}\end{cases}$$

This corresponds to an element $\mathscr{L}_{BDP}\in(\Lambda_K^-)^{ur}:=\mathbb{Z}_p^{ur}[\![\Gamma_K^-]\!]$. In the case p is split, this was done [BDP13] in the case $N^-=1$, [HB15] for general N^- (and [LZZ18] over totally real fields). In the case p is non-split, this was done by [AI19].[15]

Finite order characters $\chi\colon\Gamma_K^-\to\overline{\mathbb{Z}_p}^\times$ are now *outside* the interpolation range, but one can prove a p-adic Gross–Zagier formula. In this sense, $\mathscr{L}_{BDP}$ is still capturing the information of $L'(E/K,1)$ via Gross–Zagier, and more generally Yuan–Zhang–Zhang [YZZ13].

Theorem 7.11 (BDP formula, [HB15, Proposition 8.13])**.** *We have*

$$\mathrm{triv}(\mathscr{L}_{BDP})=e_p(1)\cdot\log_{\omega_E}y_K^{N^+,N^-}$$

where $y_K^{N^+,N^-}$ is a certain generalized Heegner point on $E(K)$, and $\log_{E(K_v)}$ is the formal group logarithm. There is a similar formula for other finite order characters.

Remark 7.12. In the case $N^-=1$, the above y_K is the usual Heegner point in $E(K)$, and the above logarithm can be identified with the logarithm on the formal group associated to E.

Now assume p is split. The characters $\chi\in\Xi_{BDP}$ have Hodge–Tate weights <-1 at v and >1 at $\overline{v}$. So $H^1_f(K_{\overline{v}},V(\chi))=0$, while $H^1_f(K_v,V(\chi))=H^1(K_v,V(\chi))$. Now consider

Definition 7.13. Let $S_{?_1,?_2}(K_\infty^?,E)$ for $?_1,?_2\in\{\mathrm{Gr},\emptyset,0\}$ denote the Iwasawa theoretic Selmer groups where the local condition at v is given by $?_1$, and at $\overline{v}$ by $?_2$. Here $\emptyset$ means no condition, and 0 means the strict condition. We also consider $S^0_{?_1,?_2}(K_\infty,E)$ having strict local condition for $w\nmid p$.

Then we expect

Conjecture 7.14 (BDP anticyclotomic main conjecture)**.** *$X^0_{\emptyset,0}(K_\infty^{anti},E)$ is Λ_K^--torsion, and its characteristic ideal is given by $(\mathscr{L}_{BDP})^2$ in $(\Lambda_K^-)^{ur}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.*

[15] [Kri21] also has a construction of a p-adic L-function in the non-split case, but it lacks an interpolation formula as above, so at the moment we cannot compare them.

It is also worth pointing out that there is a two-variable version of this. Such p-adic L-function $\mathscr{L}_{K_\infty,BDP} \in (\Lambda_K)^{ur}$ interpolates special values for $\chi\colon \Gamma_K \to \overline{\mathbb{Q}_p}^\times$ associated to unramified Hecke characters of infinity type (n,m) where $n \geq 1, m \leq -1$, and $n, m \equiv 0 \mod (p-1)$. We expect

Conjecture 7.15 (BDP two-variable main conjecture)**.** *$X_{\emptyset,0}(K_\infty, E)$ is Λ_K-torsion, and its characteristic ideal is given by $\mathscr{L}_{K_\infty,BDP}$ in $(\Lambda_K)^{ur} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Similarly as before, a control theorem assuming the BDP anticyclotomic main conjecture would give the order of $H^1_{\emptyset,0}(K,W)$, in terms of $\log y_K$. With some work, this gives $E(K)/\mathbb{Z} \cdot y_K^{n^+,N^-}$, in terms of $Ш(E/K)[p^\infty]$, which by Gross–Zagier or Yuan–Zhang–Zhang gives the p part of the BSD formula in rank 1. See [JSW17] for precise results on this.

8. Relation with Euler systems

8.1. **Perrin–Riou regulator maps.** From the exact sequence $0 \to \mathbb{Q}_p \to B_{crys}^{\phi=1} \to B_{dR}/B_{dR}^+ \to 0$, we get for a de Rham V and a padic field F that

$$0 \to V^{G_F} \to D_{crys}^{\phi=1}(V) \to D_{dR}(V)/D_{dR}^+(V) \xrightarrow{\exp_V} H^1_e(F,V) \to 0.$$

Now assume that $D_{crys}^{\phi=1}(V) = 0$. This also implies that $H^1_e(F,V) = H^1_f(F,V)$. Then the inverse of the above map is the *Bloch–Kato logarithm*

$$\log_V\colon H^1_f(F,V) \xrightarrow{\sim} \frac{D_{dR}(V)}{D_{dR}^+(V)}.$$

Moreover, if also $D_{crys}^{\phi=1}(V^*(1)) = 0$, then by dualizing the map $\exp_{V^*(1)}$ we obtain

$$\exp_V^*\colon H^1_{/f}(F,V) \xrightarrow{\sim} D_{dR}^+(V).$$

If F_∞/F is a Lubin–Tate extension, V is crystalline and has non-negative Hodge–Tate weights, then Perrin–Riou and others[16] proved that $H^1_{Iw}(F,T)/V^{G_{F_\infty}}$ is a torsion-free Λ-module of rank $\dim_{\mathbb{Q}_p} V$, and constructed a *regulator* map

$$\mathcal{L}_V\colon H^1_{Iw}(F,T) \to \mathscr{H}(\Gamma) \otimes D_{cris}(V)$$

where $\mathscr{H}(\Gamma)$ is a certain algebra of distributions, with $\Lambda \subseteq \mathscr{H}(\Gamma)$. This regulator map was defined to interpolate Bloch–Kato logarithms when specializing to $V(k)$ for $k \gg 0$ as in [PR94, Théorème], but it also interpolates Bloch–Kato dual exponentials when specializing to $V(k)$ for $k \ll 0$, as proven by Colmez.[17]

Often, one can choose suitable $\eta \in D_{cris}(V^*(1))$ so that the composition of the above with $\alpha \otimes \beta \mapsto \alpha \cdot \langle \beta, \eta \rangle$ lies in Λ. In the case of an ordinary elliptic curve $V = V_pE$ over $\mathbb{Q}_p$, V^+ and V/V^+ are of dimension 1, and in in many cases we can normalize the regulator map to obtain injections with finite cokernel

$$\mathrm{Log}\colon H^1_f(\mathbb{Q}_p, \mathbb{T}_T) \otimes_\Lambda \Lambda^{ur} \hookrightarrow \Lambda^{ur}, \quad \mathrm{Col}\colon H^1_{/f}(\mathbb{Q}_p, \mathbb{T}_T) \hookrightarrow \Lambda.$$

[16] See for example [LLZ11].
[17] See for example [Ber03].

In settings where we have Euler systems, they often afford global cohomology classes in $\mathrm{Sel}_?(F_\infty, \mathbb{T}_T)$, whose localizations are related to p-adic L-functions via these regulator maps. See also [BCD+14] for a good discussion about some cases of this. We will see some examples in what follows.

8.2. **Euler systems.** We will denote by $\mathrm{Sel}(F_\infty, E) = H^1_f(F, \mathbb{T}_T)$, with the modifications similarly to $S(F_\infty, E)$. In all this discussion, we assume that p splits in K and that p has ordinary good reduction.[18]

8.2.1. **Cyclotomic main conjecture.** In the case of $\mathbb{Q}_\infty/\mathbb{Q}$, Kato [Kat04] produced an Euler system which affords us a free rank 1 Λ-module

$$Z_{Kato} \subseteq \mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E).$$

Moreover, a deep explicit reciprocity law proven by Kato says that

Theorem 8.1 (Reciprocity law)**.** *Under the Coleman map* $\mathrm{Col}\colon H^1_{/f}(\mathbb{Q}_p, \mathbb{T}_T) \hookrightarrow \Lambda$, $\mathrm{loc}_p(Z_{Kato})$ *is sent to* $\mathscr{L}_{\mathbb{Q}_\infty, E} \cdot \Lambda$.

It is known that $\mathscr{L}_{\mathbb{Q}_\infty, E}$ is non-zero. This is how we know that Z_{Kato} is non zero. It also implies that $\mathrm{Sel}(\mathbb{Q}_\infty, E) \cap Z_{Kato} = 0$. By global duality,

$$0 \to \mathrm{Sel}(\mathbb{Q}_\infty, E) \to \mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E) \to H^1_{/f}(\mathbb{Q}_p, T) \to X(\mathbb{Q}_\infty, E) \to X_0(\mathbb{Q}_\infty, E) \to 0,$$

and we can divide by Z_{Kato}

$$0 \to \mathrm{Sel}(\mathbb{Q}_\infty, E) \to \frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)}{Z_{Kato}} \to \frac{H^1_{/f}(\mathbb{Q}_p, T)}{\mathrm{loc}_p(Z_{Kato})} \to X(\mathbb{Q}_\infty, E) \to X_0(\mathbb{Q}_\infty, E) \to 0.$$

Using this, one can prove that the cyclotomic main conjecture is equivalent to:

Conjecture 8.2 (Cyclotomic main conjecture without L-functions)**.** $\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)$ *is a rank* 1 *torsion-free* Λ*-module, and* $\mathrm{Ch}\left(\frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)}{Z_{Kato}}\right) = \mathrm{Ch}(X_0(\mathbb{Q}_\infty, E))$.

Kato proved that $X_0(\mathbb{Q}_\infty, E)$ is Λ-torsion, $\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)$ is rank 1 torsion-free and the "Euler system divisibility"

$$\mathrm{Ch}(X_0(\mathbb{Q}_\infty, E)) \text{ divides } \mathrm{Ch}\left(\frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)}{Z_{Kato}}\right)$$

using his Euler system.

Proof of equivalence. Using that $\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)$ is a rank 1 torsion-free Λ-module, we have that $\frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)}{Z_{Kato}}$, and hence $\mathrm{Sel}(\mathbb{Q}_\infty, E)$, are Λ-torsion. But $\mathrm{Sel}(\mathbb{Q}_\infty, E) \subseteq \mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)$ and the latter is torsion-free, so this means that $\mathrm{Sel}(\mathbb{Q}_\infty, E)$ is zero. From the exact sequence above, we would thus conclude that $X(\mathbb{Q}_\infty, E)$ is Λ-torsion.

Hence from Kato's result we obtain the exact sequence of torsion Λ-modules

$$0 \to \frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)}{Z_{Kato}} \to \frac{H^1_{/f}(\mathbb{Q}_p, T)}{\mathrm{loc}_p(Z_{Kato})} \to X(\mathbb{Q}_\infty, E) \to X_0(\mathbb{Q}_\infty, E) \to 0.$$

Now the equivalence of equalities of characteristic ideals follows from the reciprocity law. □

[18] There has been a lot of progress on extending these to non-split p or supersingular reduction.

More precisely, the above proof shows that Kato's divisibility translate to the divisibility

$$(\mathscr{L}_{\mathbb{Q}_\infty,E}) \text{ divides } \mathrm{Ch}(X(\mathbb{Q}_\infty,E)).$$

Remark 8.3. Skinner–Urban [SU14] adapted the techniques of Ribet and Mazur–Wiles in the context of $GU(2,2)$ to prove the opposite divisibility in the two-variable main conjecture under some technical assumptions (crucially, one of them is that $\epsilon = 1$)

$$\mathrm{Ch}(X(K_\infty,E)) \text{ divides } (\mathscr{L}_{K_\infty,E}).$$

By specializing to the cyclotomic variable, this amounts to

$$\mathrm{Ch}(X(\mathbb{Q}_\infty,E)) \cdot \mathrm{Ch}(X(\mathbb{Q}_\infty,E^K)) \text{ divides } (\mathscr{L}_{\mathbb{Q}_\infty,E}) \cdot (\mathscr{L}_{\mathbb{Q}_\infty,E^K}).$$

So in combination with Kato's result, this proves the full cyclotomic main conjecture in some cases.

8.2.2. **Anticyclotomic main conjecture.** Let's assume that $N^- = 1$ for simplicity. Then we have the Euler system of Heegner points. They are (essentially) norm compatible in the anticyclotomic tower. So we get a free rank 1 Λ-module

$$Z_{Heeg} \subseteq \mathrm{Sel}(K_\infty^{anti},E).$$

Even before the work of BDP, Perrin–Riou made the following conjecture

Conjecture 8.4 (Perrin–Riou's main conjecture)**.** *$X(K_\infty^{anti},E)$ is a rank 1 Λ-module. There is a pseudo-isomorphism $X(K_\infty^{anti},E) \sim \Lambda \oplus N \oplus N$ with $\mathrm{Ch}(N) = \mathrm{Ch}\left(\frac{\mathrm{Sel}(K_\infty^{anti},E)}{Z_{Heeg}}\right)$.*

There are analogues of this conjecture in the case $N^- \neq 1$ by using generalized Heegner points.

This conjecture can be show to be equivalent to the BDP main conjecture by a similar (in principle) but more complicated analysis as above. See [Cas17, Appendix A] for details. The crucial point is that

Theorem 8.5 (Reciprocity law, [Cas17, Theorem A.1])**.** *Under the big logarithm map $\mathrm{Log}\colon H^1_f(\mathbb{Q}_p,\mathbb{T}_T) \otimes_{\Lambda_K^-} (\Lambda_K^-)^{ur} \hookrightarrow (\Lambda_K^-)^{ur}$, we have $\mathrm{Log}(\mathrm{loc}_p(Z_{Heeg})) = \mathscr{L}_{BDP} \cdot (\Lambda_K^-)^{ur}$.*

As before, the Euler system nature of Heegner points allows one to prove the rank part and the "Euler system divisibility" (see [How04])

$$\mathrm{Ch}(N) \text{ divides } \mathrm{Ch}\left(\frac{\mathrm{Sel}(K_\infty^{anti},E)}{Z_{Heeg}}\right).$$

Remark 8.6. Xin Wan [Wan20] adapted the argument of Skinner–Urban to $GU(3,1)$ to prove the opposite divisibility in the two-variable main conjecture under some technical assumptions for the case $\epsilon = -1$. As before, this affords a proof of the full anticyclotomic main conjecture in some cases.

8.2.3. **Two variable main conjectures.** Lei–Loeffler–Zerbes [LLZ14] have constructed a free submodule

$$Z_{LLZ} \subseteq \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K_\infty,E)$$

with *two* reciprocity laws, which have (essentially) been proven in [LLZ14] and [KLZ17]: under the maps $\mathrm{Col}\colon H^1_{/f}(K_{\overline{v}},\mathbb{T}_T)\xrightarrow{\sim}\Lambda_K$ and $\mathrm{Log}\colon H^1_f(K_v,\mathbb{T}_T)\otimes_{\Lambda_K}\Lambda_K^{ur}\xrightarrow{\sim}\Lambda_K^{ur}$, we have

$$\mathrm{Col}(\mathrm{loc}_{\overline{v}}(Z_{LLZ}))=\mathscr{L}_{K_\infty,E}\cdot\Lambda_K,\quad \mathrm{Log}(\mathrm{loc}_v(Z_{LLZ}))=\mathscr{L}_{K_\infty,BDP}\cdot\Lambda_K^{ur}.$$

Conjecture 8.7 (Two variable main conjecture without L-functions)**.** $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K_\infty,E)$ *is a torsion free rank* 1 Λ_K*-modules,* Z_{LLZ} *is nonzero and* $\mathrm{Ch}(X_{\mathrm{Gr},0}(K_\infty,E))=\mathrm{Ch}\left(\frac{\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K_\infty,E)}{Z_{LLZ}}\right)$.

By arguments similar as above, given the reciprocity laws, this main conjecture is related to both of the two variable main conjectures: that $\mathrm{Ch}(X(K_\infty,E))=(\mathscr{L}_{K_\infty,E})$ and that $\mathrm{Ch}(X_{\emptyset,0}(K_\infty,E))=(\mathscr{L}_{K_\infty,BDP})$.

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