# Atkin-Lehner Theory

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## 1. Introduction and Preliminaries

The Atkin-Lehner Theory specify the structure of cusp forms of a given level $N$, separating them into *oldforms*–those that essentially come from a lower level–and *newforms*. This is a decomposition that agrees with the action of the Hecke operators. In level 1, we have a basis of cusp forms consisting of eigenforms, but this is not true anymore in level $N > 1$. Among other results, the Atkin-Lehner Theory shows that, instead, while the newforms still have a basis of eigenforms, the space of oldforms have a basis of eigenfunctions only for the anemic Hecke algebra $\mathbb{T}^{(N)}$.

In this paper, we prove the basic results of Atkin-Lehner. We prove the Main Theorem and some of its consequences, such as the Multiplicity One Theorem. Our goal is to prove the decomposition theorem

$$S_k\left(\Gamma_1\left(N\right)\right) = \bigoplus_{dM|N} \alpha_{M,N,d}\left(S_k^{\text{new}}\left(\Gamma_1\left(M\right)\right)\right).$$

We give almost a complete proof, except for a full proof of 3.11, which rely on studying the $L$-functions of newforms and their functional equations. We sketch the proof here. A full proof can be found in Miyake, T., *Modular Forms*, Theorem 4.6.19. We follow mostly Lang, S., *Introduction to Modular Forms*, but also Kani, E., *Lectures on Applications of Modular Forms to Number Theory* and Diamond, F., Shurman, J., *A First Course in Modular Forms* for some of the proofs, for instance, for the Main Theorem and the Multiplicity One Theorem.

We assume a basic understanding of modular forms of higher levels, and familiarity with Hecke operators and their action on the Fourier expansions. All modular forms considered here are for $\Gamma_1(N)$, unless otherwise specified.

We recall some definitions and facts that are not necessarily presented in a first exposition to the subject.

**Definition 1.1.** For $N \geq 1$, the modular set $X_1(N)$ is the set of pairs $(t, L)$ where $L \subset \mathbb{C}$ is a lattice and $t \in \mathbb{C}$ is a point of order $N$ with respect to $L$ (that is, $\mathbb{Z} \cdot t \cap L = N\mathbb{Z} \cdot t$) with $t$ seen modulo $L$.

**Fact.** Homogeneous functions $F\colon X_1(N) \to \mathbb{C}$, of degree $-k$ are in bijection with functions $f\colon \mathfrak{h} \to \mathbb{C}$ transforming as modular forms of weight $k$ under $\Gamma_1(N)$. The bijection is given by $F \mapsto \left(z \mapsto f(z) = F\left(\frac{1}{N}, \mathbb{Z} + \mathbb{Z}z\right)\right)$. We are going to use the functions $f$ and $F$ interchangeably.

**Definition 1.2.** The anemic Hecke algebra $\mathbb{T}^{(N)}$ is the algebra generated by $(\mathbb{Z}/N\mathbb{Z})^\times$ under the diamond action and the Hecke operators $T(n)$ for $(n, N) = 1$, acting on $S_k\left(\Gamma_1\left(N\right)\right)$. The Hecke algebra $\mathbb{T}$ drops the condition $(n, N) = 1$.

**Fact.** The Hecke algebra $\mathbb{T}^{(N)}$ is commutative and contains all its adjoints under the Petersson inner product. In fact, for prime $p$ not dividing $N$, we have $\langle p \rangle^\star = \langle p \rangle^{-1}$ and $T(p)^\star = \langle p \rangle^{-1} T(p)$, and this can be extended multiplicatively.

**Notation.** If $f$ is an eigenfunction for some Hecke algebra ($\mathbb{T}$ or $\mathbb{T}^{(N)}$), we denote its *character* $\chi_f$ by the function from the appropriate Hecke algebra to the complex numbers such that $Tf = \chi_f(T)f$ for $T$ in the appropriate Hecke algebra. By the *Dirichlet character* of an arbitrary modular form $f$, we mean the function $\epsilon\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ such that $\langle n \rangle f = \epsilon(n)f$. We extend $\epsilon\colon \mathbb{Z} \to \mathbb{C}$ by letting $\epsilon(n) = 0$ if $(n, N) > 1$.

## 2. Change of Level

We define two maps $\pi_1(d)_k, \pi_2(d)_k\colon M_k(\Gamma_1(N/d)) \to M_k(\Gamma_1(N))$ for any $d \mid N$ by first defining them as maps in the modular set $X_1(N)$: for $L$ a lattice and $t$ a point with $\mathbb{Z}\cdot t \cap L \neq \{0\}$, we define $\pi_1(d)(t,L) = (dt, L)$ and $\pi_2(d)(t,L) = (t, \frac{N}{d}\mathbb{Z}\cdot t + L)$.

**Lemma 2.1.** *$\pi_1(d)$ and $\pi_2(d)$ induce maps $\pi_1(d)_k, \pi_2(d)_k\colon M_k(\Gamma_1(N/d)) \to M_k(\Gamma_1(N))$ that maps cusp forms to cusp forms. Moreover, $\pi_1(d)_k$ is the natural injection and $\pi_2(d)_k f(z) = d^k f(dz)$.*

*Proof.* First, note that such induced maps are well-defined, since if $t$ has order $N$ in $L$, then $dt$ has order $\frac{N}{d}$ in $L$ and $t$ has order $\frac{N}{d}$ in $\frac{N}{d}\mathbb{Z}\cdot t + L$. Then:

$$\pi_1(d)_k f(z) = \pi_1(d)F\left(\tfrac{1}{N}, \mathbb{Z}+\mathbb{Z}z\right) = F\left(\tfrac{d}{N}, \mathbb{Z}+\mathbb{Z}z\right) = F\left(\tfrac{1}{N/d}, \mathbb{Z}+\mathbb{Z}z\right) = f(z) \text{ and}$$

$$\pi_2(d)_k f(z) = \pi_2(d)F\left(\tfrac{1}{N}, \mathbb{Z}+\mathbb{Z}z\right) = F\left(\tfrac{1}{N}, \tfrac{N}{d}\tfrac{1}{N}\mathbb{Z}+(\mathbb{Z}+\mathbb{Z}z)\right) = F\left(\tfrac{1}{N}, \tfrac{1}{d}\mathbb{Z}+\mathbb{Z}z\right) = d^k F\left(\tfrac{1}{N/d}, \mathbb{Z}+\mathbb{Z}dz\right) = d^k f(dz).$$

It follows that the images are holomorphic in $\mathfrak{h}$ and also that they transform as modular forms, since for $\lambda \in \mathbb{C}^\times$,

$$\pi_2(d)F(\lambda t, \lambda L) = F\left(\lambda t, \frac{N}{d}\mathbb{Z}\cdot\lambda t + \lambda L\right) = F\left(\lambda t, \lambda\left(\frac{N}{d}\mathbb{Z}\cdot t + L\right)\right) = \lambda^{-k}F\left(t, \frac{N}{d}\mathbb{Z}\cdot t + L\right) = \lambda^{-k}\pi_2(d)_k F(t,L)$$

The corresponding Fourier expansion is $\pi_2(d)_k f(q) = d^k f(q^d)$, so $\pi_2(d)_k$ maintain cusp forms. □

**Corollary 2.2.** If $d_1 d_2 \mid N$, then $\pi_i(d_1 d_2)_k = \pi_i(d_1)_k \pi_i(d_2)_k$ for $i = 1, 2$ and $\pi_1(d_1)_k$ commutes with $\pi_2(d_2)_k$.

*Proof.* The proof is clear from the formulas for $\pi_1(d)_k$ and $\pi_2(d)_k$ in the previous lemma. □

**Definition 2.3.** For $M \mid N$ and $d \mid \frac{N}{M}$, we denote by $\alpha_{M,N,d}\colon S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N))$ the composition

$$\alpha_{M,N,d} = \pi_1(N/dM)_k \circ \pi_2(d)_k.$$

Let $S_k^{\text{old}}(\Gamma_1(N)) = \sum_{dM \mid N} \operatorname{Im}(\alpha_{M,N,d})$. This is a vector subspace of $S_k(\Gamma_1(N))$, whose elements are called the *oldforms*. Let $S_k^{\text{new}}(\Gamma_1(N))$ be the orthogonal complement of $S_k^{\text{old}}(\Gamma_1(N))$ with respect to the Petersson inner product. Such elements are called the *newforms*.

*Remark* 2.4. By the previous corollary, we have that if $M_1 \mid M_2 \mid N$ with $d_1 M_1 \mid M_2$ and $d_2 M_2 \mid N$, then

$$\alpha_{M_1,N,d_1 d_2} = \alpha_{M_2,N,d_2} \circ \alpha_{M_1,M_2,d_1}.$$

**Lemma 2.5.** *If $d \mid N$, then $\pi_i(d)$ for $i = 1, 2$ commute with $\mathbb{T}^{(N)}$. Hence, $\alpha_{M,N,d}$ also commutes with $\mathbb{T}^{(N)}$.*

*Proof.* For the diamond operators, we check they commute in the level of the modular sets:

$$\begin{array}{ccc} (t,L) & \xrightarrow{\langle a\rangle} & (at,L) \\ \downarrow{\scriptstyle \pi_1(d)} & & \downarrow{\scriptstyle \pi_1(d)} \\ (dt,L) & \xrightarrow{\langle a\rangle} & (adt,L) \end{array} \qquad \begin{array}{ccc} (t,L) & \xrightarrow{\langle a\rangle} & (at,L) \\ \downarrow{\scriptstyle \pi_2(d)} & & \downarrow{\scriptstyle \pi_2(d)} \\ (t,\frac{N}{d}\mathbb{Z}\cdot t+L) & \xrightarrow{\langle a\rangle} & (at,\frac{N}{d}\mathbb{Z}\cdot t+L) \end{array}$$

The rightmost arrow in the right diagram is true because we have that $\frac{N}{d}\mathbb{Z}\cdot at + L = \frac{N}{d}\mathbb{Z}\cdot at + (L + N\mathbb{Z}\cdot t) = \frac{N}{d}\gcd(a,d)\mathbb{Z}\cdot t + L = \frac{N}{d}\mathbb{Z}\cdot t$, since $Nt \in L$ and $\gcd(a,N) = 1$.

For the action of the Hecke operators, we analyze the effect in the Fourier Expansion instead, and only, at first, for $T(p)$ for $p$ prime not dividing $N$. Let $f$ have Dirichlet character $\epsilon$. Then for $i = 1$, it follows easily since $\pi_1(d)_k$ is the indentity in the Fourier expansions, and the effect of the Hecke operators $T(n)$ in such expansions does not depend on the level if $p$ does not divide the level. For $i = 2$, we get

$$\pi_2(d)_k T(p) f(q) = \pi_2(d)_k \left( \sum_{n \geq 1} (a_{np} q^n + \epsilon(p) p^{k-1} a_n q^{pn}) \right) = d^k \sum_{n \geq 1} (a_{np} q^{nd} + \epsilon(p) p^{k-1} a_n q^{pnd}) \text{ and}$$

$$T(p) \pi_2(d)_k f(q) = d^k T(p) \left( \sum_{n \geq 1} a_n q^{nd} \right) = d^k \sum_{n \geq 1} (a_{np} q^{nd} + \epsilon(p) p^{k-1} a_n q^{pnd}),$$

so that they are equal. The result for $T(n)$ with $(n, N) = 1$ follows by induction using the multiplicativity of the $T(n)$, together with the formulas for $T(p^n)$ in function of $T(p^k)$ for $k < n$ and in function of the diamond action. □

**Corollary 2.6.** The Hecke algebra $\mathbb{T}^{(N)}$ maps newforms to newforms.

*Proof.* If $T \in \mathbb{T}^{(N)}$, $f \in S_k^{\text{old}}(\Gamma_1(N))$ and $g \in S_k^{\text{new}}(\Gamma_1(N))$, then $\langle T(g),\ f \rangle = \langle g,\ T^\star(f) \rangle$ for some $T^\star \in \mathbb{T}^{(N)}$. Since $T^\star(f) \in S_k^{\text{old}}(\Gamma_1(N))$, we have $\langle g,\ T^\star(f) \rangle = 0$, and hence $\langle T(g),\ f \rangle = 0$ for all $f$. Hence $T(g) \in S_k^{\text{new}}(\Gamma_1(N))$. □

## 3. The Main Theorem and its Consequences

**Lemma 3.1.** *There is a basis $\{f_1, \dots, f_r\}$ of $S_k(\Gamma_1(N))$ such that $f_i = \alpha_{M_i,N,d_i}(g_i)$ for some $d_i$ and $M_i$ with $d_i M_i \mid N$ and $g_i \in S_k^{new}(\Gamma_1(M_i))$, such that both the $g_i$ and the $f_i$ are eigenfunctions of $\mathbb{T}^{(N)}$.*

*Proof.* If $M \mid N$, then $\mathbb{T}^{(N)}$ maps $S_k^{\text{new}}(\Gamma_1(M))$ to itself, as $\mathbb{T}^{(N)} \subseteq \mathbb{T}^{(M)}$. Since the $\mathbb{T}^{(N)}$ are commutative normal operators, by the Spectral Theorem we have a basis of eigenfunctions for $S_k^{\text{new}}(\Gamma_1(M))$. Applying the $\alpha$ maps to such basis for each $M$, they form a generating set for $S_k(\Gamma_1(N))$, since, by 2.4, all the cusp forms are generated by the $\alpha_{M,N,d}(S_k^{\text{new}}(\Gamma_1(M)))$. Removing redundancies, we get a basis $\{f_1, \dots, f_r\}$ satisfying our requirements, since by 2.5, the $f_i$ are also eigenfunctions of $\mathbb{T}^{(N)}$. □

*Remark* 3.2. Let $T \in \mathbb{T}^{(N)}$ and $f$ be an eigenfunction of $\mathbb{T}^{(N)}$. We write $f = c_1 f_1 + \dots c_r f_r$ in the basis above. Then

$$0 = T(f) - \chi_f(T) f = \sum_{i=1}^{r} c_i (\chi_{f_i}(T) - \chi_f(T)) f_i.$$

By the linear independence of the $f_i$, we conclude that if $c_i \neq 0$, then $f$ and $f_i$ have the same character.

The following is the Main Theorem of the Atkin-Lehner Theory, which will be proved in the last section.

**Theorem 3.3** (Main Theorem)**.** *Let $f \in S_k(\Gamma_1(N))$, say $f(q) = \sum_{n \geq 0} a_n q^n$. Suppose that $a_n = 0$ if $(n, N) = 1$. Then there are $g_p \in S_k(\Gamma_1(N/p))$ for each $p \mid N$ such that $f = \sum_{p \mid N} \pi_2(p)_k(g_p)$.*

For the rest of this section we develop the consequences of this theorem.

**Theorem 3.4.** *Let $f \in S_k(\Gamma_1(N))$ be an eigenfunction of $\mathbb{T}^{(N)}$. If $a_1 = 0$, then $f \in S_k^{old}(\Gamma_1(N))$.*

*Proof.* Denote by $\lambda_n = \chi_f(T(n))$ the eigenvalue of $T(n)$. We have, for $p \nmid N$, that $0 = \lambda_p a_1 = a_p$ and also that $\lambda_p a_{p^v} = a_{p^{v+1}} + \epsilon(p)p^{k-1}a_{p^{v-1}}$. By induction we conclude $a_{p^v} = 0$ for all $v \geq 0$, and by multiplicativity this implies $a_n = 0$ for all $(n, N) = 1$. By the Main Theorem, we conclude $f$ is an oldform. □

*Remark* 3.5. In particular, if $f$ is a nonzero newform, then $a_1 \neq 0$. When $a_1 = 1$, we say $f$ is *normalized*.

**Theorem 3.6** (Multiplicity One Theorem)**.** *Let $f \in S_k^{new}(\Gamma_1(N))$ nonzero and $g \in S_k(\Gamma_1(N))$ be eigenfunctions of $\mathbb{T}^{(N)}$ with the same eigencharacter. Then $g$ is a multiple of $f$.*

*Moreover, the space $S_k^{new}(\Gamma_1(N))$ is the sum of the eigenspaces of $\mathbb{T}^{(N)}$ whose eigencharacters have multiplicity one, while $S_k^{old}(\Gamma_1(N))$ is the sum of the eigenspaces whose eigencharacters have multiplicity greater than one.*

*Proof.* Write $g = g^{\text{new}} + g^{\text{old}}$ where these are respectively the new and old parts of $g$. By 3.2, we have that $f, g, g^{\text{new}}, g^{\text{old}}$ all have the same character. So we can prove the first statement separately for $g$ being a newform and an oldform.

If $g = g^{\text{new}}$, we may assume by 3.5 that both $f$ and $g$ are normalized. Then $f - g$ would be a newform eigenfunction of $\mathbb{T}^{(N)}$ with first coeficient 0, hence must be 0 by 3.4. Hence $g$ is a multiple of $f$.

If $g = g^{\text{old}}$, write $g = \sum_{i=1}^{r} c_i \pi_1(d_i)\pi_2(d_i')g_i$, for $g_i$ newforms eigenfunctions of level $\frac{N}{d_i d_i'} < N$ and $c_i \neq 0$. If $r > 0$, let $h = \pi_1(d_1 d_1')g_1$, which has the same character of $f$ by 3.2. By 3.5, we may choose a constant $c$ such that $a_1(f - ch) = 0$. Then this means that $f - ch$, being an eigenfunction, is an oldform. Since $h$ is an oldform, then so is $f$. But this means that $f = 0$, a contradiction which forces $r = 0$, hence $g = 0$ in the first place.

For the second statement, we note that an oldform eigenfunction always has eigencharacter of multiplicity greater than one, since if $g$ has level $\frac{N}{d}$ for $d > 1$, then $\pi_1(d)g$ and $\pi_2(d)g$ are linearly independent but share the same eigencharacter. Moreover, by the first statement, the eigenspaces containing a newform are one dimensional. Since the eigenspaces of $\mathbb{T}^{(N)}$ generate $S_k(\Gamma_1(N))$ by 3.1, we conclude the second statement. □

**Corollary 3.7.** If $f \in S_k^{\text{new}}(\Gamma_1(N))$ is an eigenfunction of $\mathbb{T}^{(N)}$, then $f$ is an eigenform for the entire $\mathbb{T}$.

*Proof.* For any $T \in \mathbb{T}$, we have that $Tf$ is also a newform which is eigenfunction for $\mathbb{T}^{(N)}$. Moreover, it shares the same character of $f$ by the commutativity of $\mathbb{T}$. Hence $Tf$ is a multiple of $f$ by the previous theorem. Since this holds for any $T \in \mathbb{T}$, we conclude $f$ is an eigenform for all of $\mathbb{T}$. □

**Definition 3.8.** We denote by $\mathcal{N}(N)$ the normalized newforms of level $N$ eigenfunctions for $\mathbb{T}^{(N)}$, and hence for $\mathbb{T}$.

*Remark* 3.9. By 3.1, $\mathcal{N}(N)$ generate $S_k^{\text{new}}(\Gamma_1(N))$. By 3.6, its elements have different characters, so they form a basis of $S_k^{\text{new}}(\Gamma_1(N))$. Indeed, if we had a minimal linear dependence $\sum_{f \in \mathcal{N}(N)} c_f f = 0$ with $c_g \neq 0$, we can apply $T - \chi_g(T)$ to it, and get $\sum_{f \in \mathcal{N}(N)} c_f(\chi_f(T) - \chi_g(T))f = 0$, and so, since we chose a minimal linear dependence, we must have $c_f(\chi_f(T) - \chi_g(T)) = 0$ for all $f$ and $T$, and hence $c_f = 0$ for all $f$ in the first place.

**Lemma 3.10.** *For $f \in \mathcal{N}(M)$ and $M \mid N$, we denote $S_f(N) = \sum_{d \mid N/M} \mathbb{C} \cdot \alpha_{M,N,d}(f)$. Then, in fact, we have $S_f(N) = \bigoplus_{d \mid N/M} \mathbb{C} \cdot \alpha_{M,N,d}(f)$, that is, the $\alpha_{M,N,d}(f)$ are linearly independent.*

*Proof.* Write $f(q) = \sum_{n\geq 1} a_n q^n$ and let $n' \geq 1$ be the first index with $a_{n'} \neq 0$. Assume by contradiction that $\sum_{d|N/M} c_d \alpha_{M,N,d}(f) = 0$ with not all the $c_d$ equal to 0. Let $d'$ be the smallest index such that $c_{d'} \neq 0$. Then we have the Fourier expansion $S = \sum_{d|N/M} c_d d^k f(q^d) = 0$. Since the exponents of $q$ in $f(q^d)$ start with $dn'$, we have that the smallest exponent of $q$ in $S$ is $d'n'$, and such term comes only from $c_{d'}(d')^k f(q^{d'})$. Hence, the coefficient of $q^{d'n'}$ in $S$ is $c_{d'}(d')^k a_{d'} \neq 0$, which is a contradiction. □

Note that the elements of $S_f(N)$ share the same $\mathbb{T}^{(N)}-$characters by 2.5. In fact, $S_f(N)$ is precisely the eigenspace of $f$ under $\mathbb{T}^{(N)}$. This is a corollary of the following deep theorem, the proof of which we only sketch.

**Theorem 3.11.** *If $f \in \mathcal{N}(M_1),\ g \in \mathcal{N}(M_2)$ for $M_1, M_2 \mid N$ and they share the same eigencharacter for $\mathbb{T}^{(N)}$, then $M_1 = M_2$ and $f = g$.*

*Sketch of Proof.* Since $f$ and $g$ are newforms, they are eigenfunctions for all of $\mathbb{T}$ by 3.7, so one can prove that their $L-$series satisfy an Euler product and a functional equation similar to the case of level 1. Since they share the same eigencharacter away from $N$, we have

$$\frac{L(f,s)}{L(g,s)} = \left(\frac{M_1}{M_2}\right)^{s/2} \prod_{p|N} \frac{1-\chi_g(p)p^{-s}+\epsilon_g(p)p^{k-1-2s}}{1-\chi_f(p)p^{-s}+\epsilon_f(p)p^{k-1-2s}}$$

for $\Re(s) > \frac{k}{2}+1$, and hence on the entire plane since both sides are meromorphic. The functional equation then gives

$$\left(\frac{M_1}{M_2}\right)^{s/2} \prod_{p|N} \frac{1-\chi_g(p)p^{-s}+\epsilon_g(p)p^{k-1-2s}}{1-\chi_f(p)p^{-s}+\epsilon_f(p)p^{k-1-2s}} = c\left(\frac{M_1}{M_2}\right)^{(k-s)/2} \prod_{p|N} \frac{1-\overline{\chi_g(p)}p^{s-k}+\overline{\epsilon_g(p)}p^{2s-k-1}}{1-\overline{\chi_f(p)}p^{s-k}+\overline{\epsilon_f(p)}p^{2s-k-1}},$$

for a constant $c$, which rearranges to

$$\left(\frac{M_1}{M_2}\right)^{s} \prod_{p|N} \frac{1-\chi_g(p)p^{-s}+\epsilon_g(p)p^{k-1-2s}}{1-\chi_f(p)p^{-s}+\epsilon_f(p)p^{k-1-2s}} = c\left(\frac{M_1}{M_2}\right)^{k/2} \prod_{p|N} \frac{1-\overline{\chi_g(p)}p^{s-k}+\overline{\epsilon_g(p)}p^{2s-k-1}}{1-\overline{\chi_f(p)}p^{s-k}+\overline{\epsilon_f(p)}p^{2s-k-1}}.$$

Since both sides are Dirichlet series, the coefficients of $n^{-s}$ on both sides must be equal for each $n$. Hence

$$\left(\frac{(M_1)_p}{(M_2)_p}\right)^{s} \frac{1-\chi_g(p)p^{-s}+\epsilon_g(p)p^{k-1-2s}}{1-\chi_f(p)p^{-s}+\epsilon_f(p)p^{k-1-2s}} = c_p \frac{1-\overline{\chi_g(p)}p^{s-k}+\overline{\epsilon_g(p)}p^{2s-k-1}}{1-\overline{\chi_f(p)}p^{s-k}+\overline{\epsilon_f(p)}p^{2s-k-1}}$$

for each $p \mid N$, where $(M_i)_p$ denotes the $p-$part of $M_i$ and $c_p$ is some constant. Analyzing these equalities for all $p$ and with some more analytic input about the absolute values of $\chi_f(p)$ and $\chi_g(p)$, one can conclude that $M_1 = M_2$ and that $\chi_f(p) = \chi_g(p)$ for all $p$, and hence that $\chi_f = \chi_g$. Then, by 3.6, this implies $f = g$. □

**Corollary 3.12.** Let $f \in \mathcal{N}(M)$ for $M \mid N$. Then $S_f(N)$ is the $\mathbb{T}^{(N)}-$eigenspace defined by $f$.

*Proof.* Denote by $\mathbb{T}(\chi)$ the $\mathbb{T}^{(N)}-$eigenspace of eigencharacter $\chi$. Then we have, by 3.1,

$$S_k(\Gamma_1(N)) = \sum_{f\in\mathcal{N}(M),\ M|N} S_f(N) \subseteq \sum_{f\in\mathcal{N}(M),\ M|N} \mathbb{T}(\chi_f) \subseteq \bigoplus_{\chi} \mathbb{T}(\chi) = S_k(\Gamma_1(N)).$$

So we must have equality in all $\subseteq$. However, by the theorem above we have that all $\chi_f$ are distinct, which implies all $S_f(N) \subseteq \mathbb{T}(\chi_f)$ are equalities, and hence that $S_f(N)$ is the full eigenspace defined by $f$ for each $f \in \mathcal{N}(M)$. □

**Corollary 3.13** (Decomposition Theorem)**.**

$$S_k(\Gamma_1(N)) = \bigoplus_{dM|N} \alpha_{M,N,d}\left(S_k^{\text{new}}(\Gamma_1(M))\right).$$

*Proof.* Doing the eigenspace decomposition of $S_k(\Gamma_1(N))$ by $\mathbb{T}^{(N)}$, we get

$$S_k(\Gamma_1(N)) = \bigoplus_{f\in\mathcal{N}(M),\ M\mid N} S_f(N).$$

By 3.10 and 3.9, this becomes

$$S_k(\Gamma_1(N)) = \bigoplus_{f\in\mathcal{N}(M),\ M\mid N}\left(\bigoplus_{d\mid N/M}\mathbb{C}\cdot\alpha_{M,N,d}(f)\right) = \bigoplus_{dM\mid N}\left(\bigoplus_{f\in\mathcal{N}(M)}\mathbb{C}\cdot\alpha_{M,N,d}(f)\right) = \bigoplus_{dM\mid N}\alpha_{M,N,d}\left(S_k^{\mathrm{new}}(\Gamma_1(M))\right).$$

□

# 4. Proof of the Main Theorem

We recall the statement of the Main Theorem:

**Theorem** (Main Theorem - Version 1). *Let $f\in S_k(\Gamma_1(N))$, say $f(q)=\sum_{n\geq 0}a_nq^n$. Suppose that $a_n=0$ if $(n,N)=1$. Then there are $g_p\in S_k(\Gamma_1(N/p))$ for each $p\mid N$ such that $f=\sum_{p\mid N}\pi_2(p)_k(g_p)$.*

The first step is changing congruence subgroups from $\Gamma_1(N)$ to $\Gamma^1(N)$. We note that the elements $\alpha_d=\left(\begin{smallmatrix} d & 0\\ 0 & 1\end{smallmatrix}\right)$ satisfy

$$\pi_2(d)_k f = d\,[\alpha_d]_k\, f.$$

One can check easily that $\alpha_N\Gamma_1(N)\alpha_N^{-1}=\Gamma^1(N)$. Hence, for all $N$ we have the corresponding isomorphisms $N^{k-1}\left[\alpha_N^{-1}\right]_k : S_k(\Gamma_1(N))\to S_k\left(\Gamma^1(N)\right)$, acting in the Fourier expansions as

$$\sum a_nq^n \mapsto \sum a_nq_N^n$$

where $q_N=q^{1/N}=e^{2\pi iz/N}$. Let the map $\iota_d$ take $f(z)$ to $f(dz)$. Then, for $N=dM$, we have the commutative diagram

$$\begin{array}{ccc} S_k(\Gamma_1(M)) & \xrightarrow{\iota_d} & S_k(\Gamma_1(N)) \\ \Big\downarrow M^{k-1}\left[\alpha_M^{-1}\right]_k & & \Big\downarrow N^{k-1}\left[\alpha_N^{-1}\right]_k \\ S_k\left(\Gamma^1(M)\right) & \xrightarrow{\text{incl}} & S_k\left(\Gamma^1(N)\right) \end{array}$$

and in particular, for $d=p,\ M=\frac{N}{p}$, the Main Theorem becomes:

**Theorem** (Main Theorem - Version 2). *Let $f\in S_k\left(\Gamma^1(N)\right)$, say $f(q_N)=\sum_{n\geq 0}a_nq_N^n$. Suppose that $a_n=0$ if $(n,N)=1$. Then there are $g_p\in S_k\left(\Gamma^1(N/p)\right)$ for each $p\mid N$ such that $f=\sum_{p\mid N}g_p$.*

Now we want to translate this statement to linear algebra. For a $d\mid N$, we consider the congruence subgroup

$$\Gamma_d=\Gamma_1(N)\cap\Gamma^0(N/d)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : \frac{N}{d}\mid b,\ N\mid c,\ a\equiv d\equiv 1 \mod N\right\}.$$

One can easily check that a set of representatives for $\Gamma(N)\backslash\Gamma_d$ is $\left\{\beta_b=\left(\begin{smallmatrix} 1 & bN/d\\ 0 & 1\end{smallmatrix}\right) : 0\leq b<d\right\}$.

We consider the following projection to $S_k(\Gamma_d)$, which is a trace operator.

$$\begin{array}{cccc} \pi_d & : & S_k(\Gamma(N)) & \to & S_k(\Gamma(N)) \\ & & f & \mapsto & \frac{1}{d}\sum_{b=0}^{d-1}[\beta_b]_k\, f \end{array}$$

**Lemma 4.1.** $\pi_d\left(\sum_{n\geq 1}a_nq_N^n\right)=\sum_{n:\ d\mid n}a_nq_N^n$.

*Proof.* We have

$$\pi_d\left(\sum_{n\geq 1}a_nq_N^n\right)=\frac{1}{d}\sum_{b=0}^{d-1}\sum_{n\geq 1}a_ne^{2\pi i(z+bN/d)n/N}=\frac{1}{d}\sum_{n\geq 1}\sum_{b=0}^{d-1}a_nq_N^ne^{2\pi ibn/d}=\frac{1}{d}\sum_{n\geq 1}a_nq_N^n\sum_{b=0}^{d-1}e^{2\pi ibn/d}=\sum_{n:\ d\mid n}a_nq_N^n.$$

□

**Corollary 4.2.** If $d_1 d_2 \mid N$, then $\pi_{d_1}$ and $\pi_{d_2}$ commute.

We want to define the projection $\pi$ that preserves only the part of $f$ away from $N$, that is,

$$\pi(f) = \sum_{n \geq 1} a_n q_N^n \mapsto \sum_{n:\ (n,N)=1} a_n q_N^n.$$

By the Inclusion-Exclusion principle, such projection is

$$\pi = \prod_{p \mid N} (1 - \pi_p).$$

This allows us to rephrase the hypotheses of the Main Theorem as $f \in S_k\left(\Gamma^1(N)\right) \cap \mathrm{Ker}(\pi)$.

**Lemma 4.3.** $\mathrm{Ker}(\pi) = \sum_{p \mid N} \mathrm{Im}(\pi_p)$.

*Proof.* If we have two commuting projections $\alpha$ and $\beta$, then $\mathrm{Ker}(\alpha\beta) = \mathrm{Ker}(\alpha) + \mathrm{Ker}(\beta)$. In fact, $\supseteq$ is obvious, and if $x \in \mathrm{Ker}(\alpha\beta)$, then we write $x = y + z$ where $y = \beta x,\ z = x - y$, and we have $z \in \mathrm{Ker}(\beta)$ and $y \in \mathrm{Ker}(\alpha)$.

In particular, since $1 - \pi_p$ are projections,

$$\mathrm{Ker}(\pi) = \sum_{p \mid N} \mathrm{Ker}(1 - \pi_p) = \sum_{p \mid N} \mathrm{Im}(\pi_p).$$

□

As $\mathrm{Im}(\pi_p) = S_k(\Gamma_p)$, we can reformulate the Main Theorem as the $\subseteq$ in the following theorem. (the $\supseteq$ is trivial)

**Theorem** (Main Theorem - Version 3)**.**

$$S_k\left(\Gamma^1(N)\right) \cap \sum_{p \mid N} S_k(\Gamma_p) = \sum_{p \mid N} S_k\left(\Gamma^1(N/p)\right).$$

By considering the action of $G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ on the complex vector space $S_k(\Gamma(N))$, we can use representation theory to solve this question. Let $N = \prod_{i=1}^n p_i^{e_i}$, so that $G = \prod_{i=1}^n G_i$ where $G_i = \mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$. We define

$$H_i = \Gamma^1(p_i^{e_i})/\Gamma(p_i^{e_i}) \qquad L_i = \Gamma^1(p_i^{e_i-1})/\Gamma(p_i^{e_i}) \qquad K_i = \left(\Gamma_1(p_i^{e_i}) \cap \Gamma^0(p_i^{e_i-1})\right)/\Gamma(p_i^{e_i})$$

subgroups of $G_i$ in order that

$$S_k(\Gamma^1(N)) = S_k(\Gamma(N))^H \qquad S_k(\Gamma^1(N/p_i)) = S_k(\Gamma(N))^{\prod_{j \neq i} H_j \times L_i} \qquad S_k(\Gamma_{p_i}) = S_k(\Gamma(N))^{K_i},$$

where $H = \prod_{i=1}^n H_i$. Note this is true since we constructed $H_i, K_i$ and $L_i$ precisely such that

$$\Gamma^1(N)/\Gamma(N) = H \qquad \Gamma^1(N/p_i)/\Gamma(N) = \prod_{j \neq i} H_j \times L_i \qquad \Gamma_{p_i}/\Gamma(N) = K_i.$$

**Lemma 4.4.** $L_i = \langle H_i, K_i \rangle$.

*Proof.* $\supseteq$ is clear. We write $p = p_i,\ e = e_i$ for this proof. Let $\langle H_i, K_i \rangle = R$ and $l \in L_i$. We will replace $l$ repeatedly by an element of $RlR$ until it becomes an element of $R$. Write $l = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

We first eliminate the cases $p \mid ad$ (which only can happen when $e = 1$). If $p \mid a$, then $p$ does not divide $b$, and them $l' = l\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ does not have $p \mid a'$. Similarly, if $p \mid d$, then $p$ does not divide $c$, so $l' = l\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ suffices.

Now we reduce to $p^e \mid b, c$. For $b$, this is done by left multiplying by $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ for $\beta \equiv -bd^{-1} \mod p^e$. For $c$, this is done by right multiplying by $\left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right)$ for $\gamma \equiv -cd^{-1} \mod p^e$.

So now we have $p^e \mid b, c$ and $a \equiv d \equiv 1 \mod p^{e-1}$. As $\det(l) = 1$, we have $ad \equiv 1 \mod p^e$. Now consider the matrix

$$\gamma = \begin{pmatrix} 1 & 1-a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} a + a(1-ad) & 1-ad \\ ad-1 & d \end{pmatrix} \equiv l \mod p^e.$$

Each term of the left product belongs to $R$, so $\gamma \in R$. Since $l\gamma^{-1} \equiv 1 \mod p^e$, we get $l' = l\gamma^{-1} \in R$, as we wanted. □

**Corollary 4.5.** $\prod_{j \neq i} H_j \times L_i = \langle H, K_i \rangle$.

Hence, the Main Theorem becomes:

**Theorem** (Main Theorem - Version 4)**.** *Let $H$ and $K_i$ as above. Then*

$$S_k(\Gamma(N))^H \cap \sum_{p_i \mid N} S_k(\Gamma(N))^{K_i} = \sum_{p_i \mid N} S_k(\Gamma(N))^{\langle H, K_i \rangle}.$$

By decomposing $S_k(\Gamma(N))$ into irreducible representations of $G$, the Main Theorem follows from:

**Theorem 4.6.** *Let $V$ be an irreducible representation of a group $G = \prod_{i=1}^n G_i$, with subgroups $H = \prod_{i=1}^n H_i$ and $K = \prod_{i=1}^n K_i$. Then*

$$V^H \cap \sum_{i=1}^n V^{K_i} = \sum_{i=1}^n V^{\langle H, K_i \rangle}.$$

*Proof.* If we denote $V^{\langle H_i, K_i \rangle} = W_i$, then we may write $V^{H_i} = W_i \oplus U_i$ and $V^{K_i} = W_i \oplus T_i$. Then $U_i$ and $T_i$ are linearly disjoint, since we have $U_i \cap T_i \subseteq V^{H_i} \cap V^{K_i} = V^{\langle H_i, K_i \rangle}$. Since the representation theory of $G$ is the tensor product of the representation theory of the $G_i$, we have $V = \otimes_{i=1}^n V_i$ for irreducible representations of $G_i$ and

$$V^H = \bigotimes_{i=1}^n W_i \oplus U_i, \qquad V^{K_i} = V_1 \otimes \cdots \otimes (W_i \oplus T_i) \otimes \cdots \otimes V_n,$$
$$V^{\langle H, K_i \rangle} = (W_1 \oplus U_1) \otimes \cdots \otimes W_i \otimes \cdots \otimes (W_n \oplus U_n).$$

Now we can choose basis $b_i^1, \ldots, b_i^{r_i}$ for each $V_i$ such that we can choose basis for $U_i, W_i, T_i$ as subsets of such basis. Then a basis for $V$ is $\{b_1^{a_1} \otimes \cdots \otimes b_r^{a_r} : 1 \leq a_i \leq r_i\}$. In this way, all of $V^H$, $\sum_{i=1}^n V^{K_i}$ and $\sum_{i=1}^n V^{\langle H, K_i \rangle}$ have basis as subsets from this basis, namely:

$$V^H = \operatorname{span}(\{b_1^{a_1} \otimes \cdots \otimes b_r^{a_r} : b_i^{a_i} \in W_i \oplus U_i\}),$$
$$\sum_{i=1}^n V^{K_i} = \operatorname{span}(\{b_1^{a_1} \otimes \cdots \otimes b_r^{a_r} : \text{ for some } i,\ b_i^{a_i} \in W_i \oplus T_i\}),$$
$$\sum_{i=1}^n V^{\langle H, K_i \rangle} = \operatorname{span}(\{b_1^{a_1} \otimes \cdots \otimes b_r^{a_r} : b_i^{a_i} \in W_i \oplus U_i \text{ and for some } i,\ b_i^{a_i} \in W_i\}).$$

Now it is clear that the intersection of the first two is the third one, given that $(W_i \oplus U_i) \cap (W_i \oplus T_i) = W_i \oplus (U_i \cap T_i) = W_i$, since $W_i, U_i$ and $T_i$ are pairwise disjoint. □

I pledge my honour that this paper represents my own work in accordance with University regulations.