Hecke Eigenforms

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1. Introduction

Hecke's fundamental discovery about modular forms is that the space $M_k(\Gamma_1)$ of modular forms of weight $k > 0$ for $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ is spanned in a unique way by modular forms that are eigenfunctions of all Hecke operators simultaneously. These *Hecke eigenforms*, moreover, form a bridge to other areas of mathematics, such as number theory and algebraic geometry.

Definition 1.1. Let

$$f(z) = \sum_{n\geq 0} c(n)q^n \in M_k(\Gamma_1)$$

be a (nonzero) modular form of weight $k > 0$. We call f a *Hecke eigenform* if it is an eigenfunction for all Hecke operators $T(m)$. That is, if there are complex numbers $\lambda(m) \in \mathbb{C}$ such that

$$T(m)f = \lambda(m)f$$

for all $m \in \mathbb{Z}_{>0}$. Moreover, we call it a *normalized Hecke eigenform* if it also satisfies $c(1) = 1$.

Example 1.2. Bryan proved that E_k is a Hecke eigenform with eigenvalues $\sigma_{k-1}(n)$ for $T(n)$, with corresponding normalized Hecke eigenform being

$$\mathbb{G}_k = (-1)^{k/2}\frac{B_{k/2}}{2k}E_k = (-1)^{k/2}\frac{B_{k/2}}{2k} + \sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

Theorem 1.3. *Let f be a Hecke eigenform, and let the notation be as above. Then:*

(a) $c(1) \neq 0$.

(b) *If f is also normalized (that is, $c(1) = 1$), then*

$$\lambda(n) = c(n).$$

Proof. Bryan proved that

$$T(n)f(z) = \sum_{m\geq 0}\gamma_n(m)q^m$$

for

$$\gamma_n(m) = \sum_{1\leq a|(n,m)} a^{k-1}c\left(\frac{nm}{a^2}\right),$$

and since f is a Hecke eigenform, we know that $\gamma_n(m) = \lambda(n)c(m)$. In particular, for $m = 1$ this amounts to $c(n) = \lambda(n)c(1)$. As f is nonzero, me must have $c(1) \neq 0$, as otherwise all $c(n)$ would be zero. Moreover, if f is normalized, then this also amounts to $c(n) = \lambda(n)$. ∎

Corollary 1.4. If f, g are two normalized Hecke eigenforms with same eigenvalues $\lambda(n)$ for all $n > 0$, then $f = g$.

Proof. By the previous theorem 1.3, the eigenvalues $\lambda(n)$ determine the coefficients of the Fourier expansion, so they determine the normalized Hecke eigenform. □

Corollary 1.5. Let f be a normalized Hecke eigenform. Then we have the identities:

$$\begin{cases} c(m)c(n) = c(mn) & \text{if} \quad (m,n) = 1. \\ c(p)c(p^n) = c(p^{n+1}) + p^{k-1}c(p^{n-1}) & \text{if} \quad p \text{ is prime and } n \geq 1. \end{cases}$$

Proof. This follows directly by comparing the coeficient of q in the following identities proved by Bryan:

$$\begin{cases} T(m)T(n)f = T(mn)f & \text{if} \quad (m,n) = 1. \\ T(p)T(p^n)f = T(p^{n+1})f + p^{k-1}T(p^{n-1})f & \text{if} \quad p \text{ is prime and } n \geq 1. \end{cases}$$

□

2. Analytic Perspective on Hecke Eigenforms

These properties on the coefficients of a Hecke eigenform f can be translated to an Euler factorization-type formula for some Dirichlet series $L(f, s) = \Phi_f(s)$ associated with f.

Definition 2.1. If f is a modular form, we associate to f its *Hecke L-series* $L(f, s) = \Phi_f(s)$ defined by

$$L(f, s) = \Phi_f(s) = \sum_{n \geq 1} \frac{c(n)}{n^s}.$$

Remark 2.2. At a first moment, it is not clear where this Dirichlet series converge, or whether it converges anywhere at all. However, by a theorem of Hecke, for any modular form f we have $c(n) = O(n^{k-1})$, [1] and hence such L-series actually converges absolutely at least for $\Re(s) > k$.

Example 2.3. For $\mathbb{G}_k$, we have $c(n) = \sigma_{k-1}(n)$ for $n \geq 1$, so

$$L(\mathbb{G}_k, s) = \sum_{n \geq 1} \frac{\sigma_{k-1}(n)}{n^s} = \sum_{a,d \geq 1} \frac{a^{k-1}}{a^s d^s} = \sum_{d \geq 1} \frac{1}{d^s} \sum_{a \geq 1} \frac{a^{k-1}}{a^s} = \zeta(s)\zeta(s - k + 1).$$

Using the Euler product of $\zeta(s)$, we may write this as

$$L(\mathbb{G}_k, s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \prod_{p \text{ prime}} \frac{1}{1 - p^{-s+k-1}} = \prod_{p \text{ prime}} \frac{1}{(1 - p^{-s})(1 - p^{-s+k-1})} = \prod_{p \text{ prime}} \frac{1}{1 - \sigma_{k-1}(p)p^{-s} + p^{k-1-2s}}.$$

In fact, this generalizes for all normalized Hecke eigenforms:

Theorem 2.4. *If f is a normalized Hecke eigenform, then we have the following Euler product-type formula:*

$$L(f, s) = \prod_{p \ prime} \frac{1}{1 - c(p)p^{-s} + p^{k-1-2s}}.$$

[1] For completeness, we include a proof of this result in the next section.

Proof. Since the function $c(n)$ is multiplicative, and converges absolutely for $\Re(s) > k$, we have the usual product formula [2]

$$L(f,s) = \prod_{p \text{ prime}} \left(\sum_{n\geq 0} \frac{c(p^n)}{p^{ns}} \right).$$

Hence we are left to prove that

$$\sum_{n\geq 0} \frac{c(p^n)}{p^{ns}} = \frac{1}{1 - c(p)p^{-s} + p^{k-1-2s}}.$$

For this, we denote $\Phi_{f,p}(s) = \sum_{n\geq 0} \frac{c(p^n)}{p^{ns}}$ and consider the expression

$$S_p(s) = -c(p)\Phi_{f,p}(s) + p^s\Phi_{f,p}(s) + p^{k-1-s}\Phi_{f,p}(s) = -c(p)\sum_{n\geq 0} \frac{c(p^n)}{p^{ns}} + \sum_{n\geq 0} \frac{c(p^n)}{p^{(n-1)s}} + p^{k-1}\sum_{n\geq 0} \frac{c(p^n)}{p^{(n+1)s}}.$$

Since it converges absolutely for $\Re(s) > k$, we can write this as

$$S_p(s) = \sum_{n\geq 1} \left(\frac{-c(p)c(p^n) + c(p^{n+1}) + p^{k-1}c(p^{n-1})}{p^{ns}} \right) - c(p) + (c(p) + p^s) = p^s,$$

where the last equality follows from the term inside the summation being 0 by 1.5. So

$$p^s = S_p(s) = (-c(p) + p^s + p^{k-1-s})\Phi_{f,p}(s),$$

which implies

$$\Phi_{f,p}(s) = \frac{1}{1 - c(p)p^{-s} + p^{k-1-2s}},$$

and hence that

$$L(f,s) = \prod_{p \text{ prime}} \frac{1}{1 - c(p)p^{-s} + p^{k-1-2s}}.$$

□

In fact, these L-series are very rich analytic objects. For instance, one have the following theorem due to Hecke:

Theorem. *If f is a (cusp) modular form of weight $k > 0$, then $L(f,s)$ extends to a (entire) meromorphic function in the whole complex plane. Moreover, the function*

$$X_f(s) = (2\pi)^{-s}\Gamma(s)L(f,s), \text{ where } \Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt,$$

satisfies the functional equation

$$X_f(s) = (-1)^{k/2}X_f(k-s).$$

In fact, Hecke proved also a converse: that if a Dirichlet series Φ satisfy some functional equation of this type and some regularity and growth conditions, then it comes from a modular form.

[2] This holds since

$$L(f,s) - \prod_{p\leq N \text{ prime}} \left(\sum_{n\geq 0} \frac{c(p^n)}{p^{ns}} \right) = \sum_{n \text{ with all prime factors} > N} \frac{c(n)}{n^s} \xrightarrow[N\to\infty]{} 0.$$

3. The Proofs

3.1. **Bounds on Fourier coefficients.** We prove the bound $c(n) = O(n^{k-1})$ for any modular form f. In fact, we start with a better bound for cusp forms:

Lemma 3.1. *Let f be a cusp form of weight $k > 0$. Then $c(n) = O(n^{k/2})$.*

Proof. We write $y = \Im(z)$. As f is a cusp form, we have $c(0) = 0$, hence $|f(z)| = O(q) = O(e^{-2\pi y})$ when $q \to 0$. Let $\phi(z) = |f(z)|y^{k/2}$. Then we have

$$\phi(z+1) = |f(z+1)|y^{k/2} = |f(z)|y^{k/2} = \phi(z)$$
$$\phi\left(-\tfrac{1}{z}\right) = |f\left(-\tfrac{1}{z}\right)|\left(\tfrac{y}{|z|^2}\right)^{k/2} = |z^k||f(z)|y^{k/2}|z|^{-k} = |f(z)|y^{k/2} = \phi(z)$$

So $\phi(z)$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$. Moreover, it is continuous in the fundamental domain, and it goes to 0 as $y \to \infty$, since

$$\phi(z) = |f(z)|y^{k/2} = y^{k/2}O(e^{-2\pi y}) = o(1).$$

Hence $\phi(z)$ is actually bounded, say $\phi(z) \le M$ for all z. Then this means $|f(z)| \le My^{-k/2}$. Now fix $y > 0$ and let C_y be the circle $x \mapsto q = e^{2\pi i(x+iy)}$. Then the residue theorem gives us

$$c(n) = \frac{1}{2\pi i}\int_{C_y} f(z)q^{-n-1}\,dq = \int_0^1 f(x+iy)q^{-n}\,dx,$$

so that $|c(n)| \le My^{-k/2}e^{2\pi ny}$. In particular, for $y = \frac{1}{n}$, this amounts to $|c(n)| \le e^{2\pi}Mn^{k/2}$. □

Theorem 3.2. *If f is a modular form of weight $k > 0$, then $c(n) = O(n^{k-1})$.*

Proof. We know that the space $M_k(\Gamma_1)$ of modular forms is spanned by the cusp forms $M_k^0(\Gamma_1)$ and by E_k, since if f is a modular form of weight k, then $f - c(0)E_k$ is a cusp form.

Hence, by the previous lemma 3.1, it suffices to prove this bound for $f = E_k$. Eric showed that

$$E_k(z) = 1 + \frac{2k(-1)^{k/2}}{B_{k/2}}\sum_{n\ge1}\sigma_{k-1}(n)q^n,$$

so that $c(n) = O(\sigma_{k-1}(n))$. Moreover, since

$$\frac{\sigma_{k-1}(n)}{n^{k-1}} = \sum_{d|n}\left(\frac{d}{n}\right)^{k-1} = \sum_{d|n}\frac{1}{d^{k-1}} \le \sum_{d\ge1}\frac{1}{d^{k-1}} = \zeta(k-1),$$

we have $\sigma_{k-1}(n) = O(n^{k-1})$, hence $c(n) = O(n^{k-1})$ as we wanted. □

3.2. **Extension of L-series.** We prove that the L-series extend meromorphically to the entire complex plane. The proof is similar to the proof that $\zeta(s)$ extends meromorphically.

Lemma 3.3.

$$(2\pi)^{-s}\Gamma(s) = n^s\int_0^\infty t^{s-1}e^{-2\pi nt}\,dt\,.$$

Proof. Let $\lambda = 2\pi n$. We have

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt = \int_0^\infty (\lambda t)^{s-1}e^{-\lambda t}\,d(\lambda t) = \lambda^s\int_0^\infty t^{s-1}e^{-\lambda t}\,dt,$$

and substituting $\lambda = 2\pi n$ gives the desired result. □

Theorem 3.4. *If f is a (cusp) modular form of weight $k > 0$, then $L(f,s)$ extends to a (entire) meromorphic function in the whole complex plane. Moreover, the function*

$$X_f(s) = (2\pi)^{-s}\Gamma(s)L(f,s)$$

satisfies the functional equation

$$X_f(s) = (-1)^{k/2}X_f(k-s).$$

Proof. We first prove this for a cusp form. We consider $X_f(s) = (2\pi)^{-s}\Gamma(s)L(f,s)$. Then, by the previous lemma:

$$X_f(s) = \sum_{n\geq 1} \frac{c(n)(2\pi)^{-s}\Gamma(s)}{n^s} = \sum_{n\geq 1} c(n)\int_0^\infty t^{s-1}e^{-2\pi nt}\,dt\,.$$

For $\Re(s) > k$, this converges absolutely, [3] so we can change the order of the summation and the integral:

$$X_f(s) = \int_0^\infty t^{s-1}\sum_{n\geq 1} c(n)e^{-2\pi nt}\,dt = \int_0^\infty t^{s-1}(f(it) - f(\infty))\,dt\,.$$

We already know that $f(it) - f(\infty)$ decays exponentially as $t \to \infty$. As f is a cusp form, $f(\infty) = 0$, and since $f\left(i\frac{1}{t}\right) = (it)^k f(it)$, $f(it)$ also decays exponentially as $t \to 0$. Hence $X_f(s)$ is holomorphic in the entire complex plane, and so

$$L(f,s) = (2\pi)^s X_f(s)\frac{1}{\Gamma(s)}$$

can also be extended holomorphically.

For the functional equation, we have

$$X_f(s) = \int_0^\infty t^{s-1}f(it)\,dt = \int_0^\infty t^{s-1}(it)^{-k}f\left(i\frac{1}{t}\right)dt = \int_0^\infty u^{k+1-s}i^{-k}f(iu)u^{-2}\,du = (-1)^{k/2}\int_0^\infty t^{k-s-1}f(iu)\,du,$$

which is precisely

$$X_f(s) = (-1)^{k/2}X_f(k-s).$$

Now, as before, we just need to prove that the modular forms $\mathbb{G}_k$ have meromorphic L-series that satisfy the functional equation. We already saw in 2.3 that $L(\mathbb{G}_k, s) = \zeta(s)\zeta(s-k+1)$, which is meromorphic.

It remains to prove it satisfy the functional equation. We denote $\tilde{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, satisfying $\tilde{\zeta}(s) = \tilde{\zeta}(1-s)$, so that

$$X_{\mathbb{G}_k}(s) = (2\pi)^{-s}\Gamma(s)L(f,s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-k+1) = \tilde{\zeta}(s)\tilde{\zeta}(s-k+1)2^{-s}\pi^{(k-1)/2}\frac{\Gamma(s)}{\Gamma(s/2)\Gamma((s-k+1)/2)}.$$

By the multiplication formula $\Gamma(2z) = \Gamma(z)\Gamma(z+\frac{1}{2})\frac{2^{2z-1}}{\sqrt{\pi}}$, [4] we may write this as

$$X_{\mathbb{G}_k}(s) = \frac{1}{2}\tilde{\zeta}(s)\tilde{\zeta}(s-k+1)\pi^{k/2-1}\frac{\Gamma((s+1)/2)}{\Gamma((s-k+1)/2)}.$$

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[3] This is true since if $l = \Re(s) > k$, $u = nt$ and $v = n$,

$$\sum_{n\geq 1}|c(n)|\int_0^\infty |t^{s-1}e^{-2\pi nt}|\,dt \leq C\int_1^\infty\int_0^\infty n^{k-1}t^{l-1}e^{-2\pi nt}\,dt\,dn = C\int_1^\infty\int_0^\infty u^{l-1}v^{k-l+1}e^{-2\pi u}v^{-1}\,du\,dv = C\cdot\Gamma(s)\int_1^\infty v^{k-l}\,dv < \infty.$$

[4] This and the following identity of Γ are proved as follows: one check both sides have the same poles with same residues, so that their difference is entire. Then one prove the difference is bounded, by checking it is invariant under some horizontal translation and that tends to 0 for $i\infty$. By Liouville's theorem, this implies the difference is a constant, and evaluation at a particular point gives the equality.

Then
$$\frac{X_{\mathbb{G}_k}(s)}{X_{\mathbb{G}_k}(k-s)} = \frac{\Gamma((s+1)/2)}{\Gamma((s-k+1)/2)} : \frac{\Gamma((k-s+1)/2)}{\Gamma((-s+1)/2)}.$$
Finally, using that $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we get
$$\frac{X_{\mathbb{G}_k}(s)}{X_{\mathbb{G}_k}(k-s)} = \frac{\sin(\pi(s+1)/2)}{\sin(\pi(s-k+1)/2)} = (-1)^{k/2}$$
as $k/2$ is an integer.

As the modular forms $M_k(\Gamma_1)$ are generated by the cusp forms $M_k^0(\Gamma_1)$ and $\mathbb{G}_k$, we conclude the validity of the theorem in the general case. □

3.3. **Basis formed by Hecke eigenforms.** We mention how one proves that the space $M_k(\Gamma_1)$ has a unique basis of normalized Hecke eigenforms.

Theorem 3.5. *The space $M_k(\Gamma_1)$ of modular forms of weight $k > 0$ has a unique basis of normalized Hecke eigenforms. Moreover, the Fourier coefficients for such forms are real numbers, and all Hecke eigenforms are a multiple of one of the elements of this basis.*

Sketch of Proof. One introduces a Hermitian positive nondegenerate scalar product (the Petersson scalar product) on the space of cusp forms $M_k^0(\Gamma_1)$ by
$$\langle f, g\rangle = \int_D f(z)\overline{g(z)}y^{k-2}\, dx\, dy\, .$$
One can check that the $T(n)$ are self-adjoint operators for this scalar product, that is,
$$\langle T(n)f, g\rangle = \langle f, T(n)g\rangle\, .$$
Then, since the operators $T(n)$ commute with each other and are self-adjoint, one can find, by linear algebra, a basis formed by simultaneous eigenfunctions for all $T(n)$. This is simply a basis of Hecke eigenforms, and we can normalize each one. Moreover, all the eigenvalues for such basis are real numbers, that is, the Fourier coefficients of such normalized Hecke eigenforms are real numbers.

Such basis of $M_k^0(\Gamma_1)$ together with $\mathbb{G}_k$ form a basis of Hecke eigenforms of $M_k(\Gamma_1)$. Also, all the Fourier coefficients in these elements are real numbers.

For the uniqueness of such basis, let $f_1, \ldots, f_n$ be a basis of normalized Hecke eigenforms, where $n = \dim(M_k(\Gamma_1))$, and suposse $f_{n+1} = \sum_{i=1}^n a_i f_i$ is another Hecke eigenform. Then if $T(m)f_i = \lambda_i(m)f_i$, we must have
$$\lambda_{n+1}(m)\cdot\left(\sum_{i=1}^n a_i f_i\right) = \lambda_{n+1}(m)f_{n+1} = T(m)f_{n+1} = T(m)\left(\sum_{i=1}^n a_i f_i\right) = \sum_{i=1}^n a_i T(m) f_i = \sum_{i=1}^n a_i\lambda_i(m)f_i.$$
Since the $f_1, \ldots, f_n$ are linearly independent, this means that $\lambda_{n+1}(m)a_i = \lambda_i(m)a_i$ for all $m \in \mathbb{Z}_{>0}$ and $1 \le i \le n$. So if $a_j \neq 0$ for some j, we conclude that $\lambda_{n+1}(m) = \lambda_j(m)$ for all $m \in \mathbb{Z}_{>0}$. Since $f_1, \ldots, f_n$ are distinct normalized Hecke eigenforms, this cannot happen for two distinct indexes j because of 1.4. As f_{n+1} is nonzero, this means that precisely one of the a_i is nonzero, and hence that f_{n+1} is some multiple of one of the f_i for $1 \le i \le n$, again by 1.4.

This also proves the last assertion, that all Hecke eigenforms are a multiple of one of the f_i. □