# Modular Symbols

MURILO CORATO ZANARELLA

November 14, 2017

## 1. Introduction

Dan proved the following proposition:

**Proposition 1.1.** For $\alpha, \beta, \gamma \in \mathfrak{h}^*$, we have

(a) $\{\alpha,\ \beta\} + \{\beta,\ \gamma\} = \{\alpha,\ \gamma\}$

(b) $\{g\alpha,\ g\beta\} = \{\alpha,\ \beta\}$ for $g \in \Gamma$.

(c) If the genus of $\Gamma\backslash\mathfrak{h}^*$ is 0, then $\{\alpha,\ \beta\} \in H_1(\Gamma\backslash\mathfrak{h}^*, \mathbb{Z})$ iff $\beta \in \Gamma\alpha$.

**Definition 1.2.** Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$. We call an element $g \in \Gamma$

- *elliptic* if it has a fixed point in $\mathfrak{h}$, that is, in the interior of the upper half plane. Such fixed points are called the *elliptic* points of $\mathfrak{h}^*$. These are the points $\alpha$ with $n_\alpha = |\mathrm{Stab}_\Gamma(\alpha)| > 1$.
- *parabolic* if it has a cusp as a fixed point.

*Remark* 1.3. For elliptic elements $g \in \Gamma$, one can show that the corresponding elliptic points must be a translation of one of the following:

- $i$, in which case $g$ is a conjugate (in $\mathrm{PSL}_2(\mathbb{Z})$) of $S$.
- $\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, in which case $g$ is a conjugate (in $\mathrm{PSL}_2(\mathbb{Z})$) of $ST$.

Moreover, such $g$ cover all the elliptic elements of $\mathrm{PSL}_2(\mathbb{Z})$.

In the parabolic case, the $g$'s are precisely the conjugates (in $\mathrm{PSL}_2(\mathbb{Z})$) of $T^r$, for some $r \in \mathbb{Z}$.

We denote by $\mathrm{X}(\Gamma)$ the set of elliptic and parabolic elements of $\Gamma$. The preceding remark implies that:

**Lemma 1.4.** $\mathrm{X}(\Gamma) = \mathrm{X}(\mathrm{PSL}_2(\mathbb{Z})) \cap \Gamma$.

**Theorem 1.5.** *Let $\alpha \in \mathfrak{h}^*$. The map*

$$\begin{array}{rccc} \phi_\alpha\colon & \Gamma & \to & H_1(\Gamma\backslash\mathfrak{h}^*,\ \mathbb{Z}) \\ & g & \mapsto & \{\alpha,\ g\alpha\} \end{array}$$

*is a surjective group homomorphism which does not depend on the choice of $\alpha$. The kernel is generated by the commutator $\Gamma'$ of $\Gamma$ and by $\mathrm{X}(\Gamma)$, that is, the elliptic and parabolic elements of $\Gamma$.*

*Proof.* The fact that $\phi_\alpha$ is a group homomorphism follows from (a) and (b) of 1.1:

$$\phi_\alpha(gh) = \{\alpha,\ gh\alpha\} \overset{(a)}{=} \{\alpha,\ g\alpha\} + \{g\alpha,\ gh\alpha\} \overset{(b)}{=} \{\alpha,\ g\alpha\} + \{\alpha,\ h\alpha\} = \phi_\alpha(g) + \phi_\alpha(h).$$

Also from 1.1 it follows that $\phi = \phi_\alpha$ is independent of $\alpha$, since if $\beta \in \mathfrak{h}^*$, then for $g \in \Gamma$,

$$\phi_\alpha(g) = \{\alpha,\ g\alpha\} \overset{(a)}{=} \{\alpha,\ \beta\} + \{\beta,\ g\beta\} + \{g\beta,\ g\alpha\} \overset{(b)}{=} \phi_\beta(g) + \{\alpha,\ \beta\} + \{\beta,\ \alpha\} \overset{(a)}{=} \phi_\beta(g) + \{\alpha,\ \alpha\} = \phi_\beta(g).$$

We denote by $\overline{\alpha}$ the projection of $\alpha \in \mathfrak{h}^*$ to $\Gamma\backslash\mathfrak{h}^*$. Let $\mathfrak{h}^0$ be the complement of $\mathrm{PSL}_2(\mathbb{Z})i \cup \mathrm{PSL}_2(\mathbb{Z})\rho$ in $\mathfrak{h}$, and let $D^0 = \Gamma\backslash\mathfrak{h}^* - \{\text{elliptic points and cusps}\}$, so that $\overline{\mathfrak{h}^0} = D^0$.

To prove the remaining assertions, we must use a geometric interpretation of this homomorphism.

Since we proved $\phi$ does not depend on $\alpha$, we can choose $\alpha \in \mathfrak{h}^0$.

We now define a surjective group homomorphism $\psi_\alpha \colon \pi_1(D^0, \overline{\alpha}) \to \Gamma$ by what follows: given a closed loop $\overline{\gamma}$ on $D^0$ starting at $\overline{\alpha}$, we can lift it uniquely to a (not necessarily closed) path $\gamma$ on $\mathfrak{h}^0$ starting at $\alpha$. [1] The endpoint of such path $\beta$ satisfies $\overline{\beta} = \overline{\alpha}$, so it is $\beta = g\alpha$ for a uniquely determined $g \in \Gamma$. Then we define $\psi_\alpha(\overline{\gamma}) = g$. It is a group homomorphism since if $\psi_\alpha(\overline{\gamma_1}) = g_1,\ \psi_\alpha(\overline{\gamma_2}) = g_2$, then a lift of $\overline{\gamma} = \overline{\gamma_1} \star \overline{\gamma_2}$ is $\gamma = \gamma_1 \star (g_1 \circ \gamma_2)$, and as

$$\gamma(1) = (g_1 \circ \gamma_2)(1) = g_1(\psi_\alpha(\overline{\gamma_2})\alpha) = g_1 g_2 \alpha,$$

we conclude $\psi_\alpha(\overline{\gamma}) = g_1 g_2 = \psi_\alpha(\overline{\gamma_1})\psi_\alpha(\overline{\gamma_2})$.

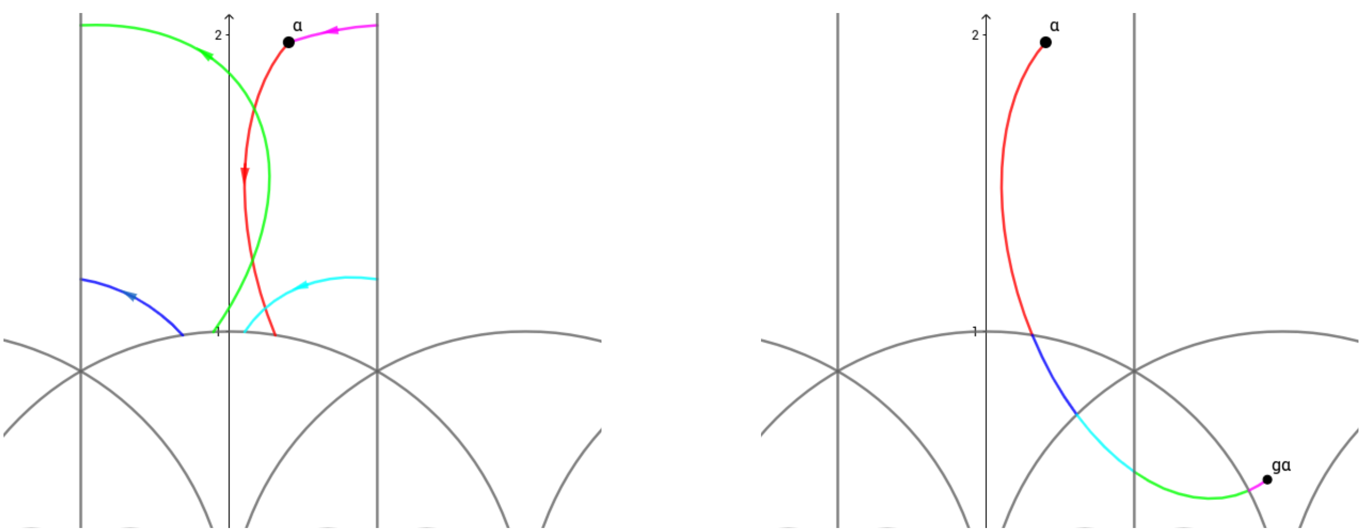


FIGURE 1. A visual description of the map $\psi_\alpha$ for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Note how the path in $\Gamma\backslash\mathfrak{h}^*$ in the left correspond to the element $g = TS \in \Gamma$.

The surjectivity of $\psi_\alpha$ follows since, for any $g \in \Gamma$, there is a path $\gamma$ from $\alpha$ to $g\alpha$ in $\mathfrak{h}^0$ (since elliptic points and cusps are a discrete set), and then $\psi_\alpha(\overline{\gamma}) = g$.

Given this description, one can see easily that the composite map

$$\tau \colon \pi_1(D^0, \overline{\alpha}) \xrightarrow{\psi_\alpha} \Gamma \xrightarrow{\phi} H_1(\Gamma\backslash\mathfrak{h}^*,\ \mathbb{Z})$$

coincides with the canonical homomorphism of the fundamental group of the surface $D^0$ into the homology group of its compactification $\Gamma\backslash\mathfrak{h}^*$. This immediately implies that the map $\phi$ is surjective. Furthermore, since $\psi_\alpha$ is also surjective,

[1]This follows since $\mathfrak{h}^0 \to D^0$ is a *covering map*.

we may write

$$\mathrm{Ker}\,(\phi) = \psi_\alpha(\mathrm{Ker}\,(\psi_\alpha \circ \phi)).$$

Since $H_1(\Gamma\backslash\mathfrak{h}^0,\ \mathbb{Z})$ is abelian, $\tau$ must factor into the abelianization of $\pi_1(D^0, \overline{\alpha})$. As $\pi_1(D^0, \overline{\alpha})^{\mathrm{ab}} \xrightarrow{\sim} H_1(D^0,\ \mathbb{Z})$, [2] we have

$$\tau\colon \pi_1(D^0, \overline{\alpha}) \to \pi_1(D^0, \overline{\alpha})^{\mathrm{ab}} \xrightarrow{\sim} H_1(D^0,\ \mathbb{Z}) \to H_1(\Gamma\backslash\mathfrak{h}^*,\ \mathbb{Z}).$$

Since $D^0 = \Gamma\backslash\mathfrak{h}^* - \{\text{elliptic points and cusps}\}$, one can see that the kernel of the last map is generated by small loops around each elliptic point or cusp. [3] For $\beta \in \{\text{elliptic points and cusps}\}$, we denote by $\gamma_\beta$ such a loop.

This discussion shows that $\mathrm{Ker}\,(\psi \circ \phi) = \pi_1(D^0, \overline{\alpha})' \langle \gamma_\beta \rangle$. Since $\psi$ is surjective, this means

$$\mathrm{Ker}\,(\phi) = \Gamma' \langle \psi(\gamma_\beta) \rangle .$$

So now it remains to prove that $\langle \psi(\gamma_\beta) \rangle = \mathrm{X}\,(\Gamma)$. For any such $\beta$, we can find a unique $g_\beta \in \mathrm{PSL}_2(\mathbb{Z})$ such that $g_\beta(\beta) \in \{i, \rho, i\infty\}$. Then $g_\beta \psi(\gamma_\beta) g_\beta^{-1} = \psi(g_\beta \circ \gamma_\beta)$, and since $g_\beta \circ \gamma_\beta$ is a small loop around one of $\{i, \rho, i\infty\}$, we can analyze those explicitly and conclude $g_\beta \psi(\gamma_\beta) g_\beta^{-1}$ is $S, TS$ or $T^r$ for some $r$ depending whether $g_\beta(\beta)$ is $i, \rho$ or $i\infty$. This implies

$$\psi(\gamma_\beta) \in \mathrm{X}\,(\mathrm{PSL}_2(\mathbb{Z})) \cap \Gamma = \mathrm{X}\,(\Gamma)\,,$$

This shows also that $\psi(\gamma_\beta)$ fix $\beta$, and this, in turn, uniquely determine $\{\psi(\gamma_\beta), \psi(\gamma_\beta)^{-1}\}$, given that it fixes $\beta$, if $\beta$ is elliptic. If $\beta$ is a cusp, we note $\psi(\gamma_\beta)$ is a conjugate (in $\mathrm{PSL}_2(\mathbb{Z})$) of $T^H$, and this, again, uniquely determine $\psi(\gamma_\beta)$ together with the fact it fixes $\beta$. Finally the parabolic elements of $\mathrm{X}\,(\Gamma)$ are the conjugates of some $T^{rH}$ that are inside $\Gamma$, for some $r \in \mathbb{Z}$, and so they are all generated by $\langle \psi(\gamma_\beta) \rangle$.

Hence $\langle \psi(\gamma_\beta) \rangle = \mathrm{X}\,(\Gamma)$, and we conclude $\mathrm{Ker}\,(\phi) = \Gamma'\ \mathrm{X}\,(\Gamma)$. □

## 2. Distinguished Classes

Let $J = \Gamma\backslash\mathrm{PSL}_2(\mathbb{Z})$ be the set of right cosets. We define the map

$$\xi\colon J \to H_1(\Gamma\backslash\mathfrak{h}^*, \mathbb{R})$$

by: if $j \in J$ and $g$ is any representative of the class $j$, let

$$\xi(j) = \{g(0),\ g(i\infty)\}\,.$$

By 1.1(b), this does not depend on the choice of representative $g$.

**Definition 2.1.** We define the classes in $\xi(J)$ as above to be the *distinguished classes*.

*Remark* 2.2. Note that the distinguished classses are not, in general, integral classes.

---

[2] This is called the *Hurewicz's theorem*.

[3] For a proof of this, one can use the Mayer-Vietoris exact sequence. Around each elliptic or cusp $\beta$ we take open sets $U_\beta$ such that $U_\beta$ are all disjoint, and let $U = \bigcup_\beta U_\beta$, $V = \Gamma\backslash\mathfrak{h}^* - \bigcup_\beta\{\beta\}$ so that $U \cap V = \bigcup_\beta(U_\beta - \{\beta\})$. Also $U \cap V$ has a retraction to $\coprod_r S^1$, where $r$ is the number of elliptic points and cusps in $\Gamma\backslash\mathfrak{h}^*$. $U$ has the same homology of $\coprod_r D^2$, and $V$ is a deformation retract of $D^0$. Finally, $U \cup V = \Gamma\backslash\mathfrak{h}^*$. By the exactness of $H_1\left(\coprod_r S^1\right) \to H_1\left(\coprod_r D^2\right) \oplus H_1\left(D^0\right) \to H_1\left(\Gamma\backslash\mathfrak{h}^*\right)$, and since $H_1\left(\coprod_r D^2\right) = 0$ we get that the kernel we want is simply $\mathrm{Im}\left(\coprod_r S^1 \to H_1\left(D^0\right)\right)$.