KUGA-SATAKE CONSTRUCTION

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Abstract. We explain the Kuga-Satake construction for Hodge structures of K3-type and how it can be used to derive the Weil and Tate conjectures for K3 surfaces defined over $\mathbb{Q}$.

Contents

We explain the Kuga-Satake construction that associates an isogeny class of abelian varieties to certain rational Hodges structures, such as the ones coming from K3 surfaces. We show how this construction can be used to transport the validity of the Weil conjecture and partial validity of the Tate conjecture from abelian varieties to their validity to K3 surfaces defined over $\mathbb{Q}$.

This idea has been used by Deligne to prove the Weil conjecture for K3 surfaces over finite fields prior to his proof for all varieties. More recently, in 2013, the Tate conjecture for K3 surfaces over finite fields of odd characteristic has been proven by Madapusi Pera also using this construction.

1. K3 surfaces

We begin by recalling the definition of K3 surfaces. The only particular property of K3 surfaces that will be used is its Hodge diamond.

Definition 1.1. A K3 surface is a smooth projective algebraic variety of dimension 2 with trivial canonical bundle K and trivial first homology.

Proposition 1.2. *Let X be a K3 surface. Then its Hodge diamond is*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proof. The zeroes follow from the triviality of the first homology together with Poincaré duality.

Since X is connected, we clearly have $h^{0,0} = 1$. Since $K = \Omega^2(X)$ is trivial, we have that $h^{2,0} = 1$. Now, by Serre duality, we conclude that $h^{0,2} = h^{2,2} = 1$. In particular, $\chi(\mathcal{O}_X) = 2$.

Although we will not need it here, $h^{1,1}$ can be computed by the Noether's formula: we have

$$\chi(\mathcal{O}_X) = \frac{K.K + \chi}{12}$$

where χ is the topological Euler characteristic. Since K is trivial, we conclude that $\chi = 24$ and hence that $h^{1,1} = 20$. □

2. Hodge and Tate structures

2.1. **Hodge structures.** We construct a category of Hodge structures. These are usually called pure Hodge structures in the literature.

Definition 2.1. A Hodge structure of weight $k \in \mathbb{Z}$ is a $\mathbb{Q}$-vector space V with a decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}, \quad \text{and } \overline{V^{p,q}} = V^{q,p}$$

where $p, q \in \mathbb{Z}$.

Proposition 2.2. *For a $\mathbb{Q}$-vector space V and an integer k, there is a bijection*

$$\left\{ \begin{array}{c} \text{algebraic representations } h\colon \mathbb{C}^\times \to \mathrm{GL}(V_{\mathbb{R}}) \\ \text{with } h(t) = t^k \text{ for } t \in \mathbb{R}^\times \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Hodge structures on } V \\ \text{of weight } k \end{array} \right\}$$

given by

$$h \longmapsto (V^{p,q} := \{v \in V_{\mathbb{C}} \colon h(z)v = z^p \overline{z}^q v\}).$$

Proof. We first prove that given such algebraic representation h, we have $V = \bigoplus_{p+q=k} V^{p,q}$. This is where we use that h is algebraic: since $\mathbb{C}^\times \subseteq \mathrm{GL}_2(\mathbb{R})$ by $z \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, $h(z)$ is a matrix whose coefficients are polynomials in $x,\ y,\ (x^2+y^2)^{-1} = (z\overline{z})^{-1}$. So if λ is an irreducible character of h, we must have $\lambda(z) = z^p \overline{z}^q$ for some $p, q \in \mathbb{Z}$, and since $\lambda(t) = t^k$ for $t \in \mathbb{R}^\times$, we must have $p + q = k$. Now if follows that $V = \bigoplus_{p+q=k} V^{p,q}$. That $V^{q,p} = \overline{V^{p,q}}$ follows at once from the fact that h is a real representation.

Conversely, given a Hodge structure $V = \bigoplus_{p+q=k} V^{p,q}$, we define h by setting $h(z)v = z^p \overline{z}^q v$ for $v \in V^{p,q}$. This is well defined since, for $v \in V_{\mathbb{R}}$, we have

$$\overline{h(z)v} = \sum_{p+q=k} \overline{z^p \overline{z}^q v^{p,q}} = \sum_{p+q=k} z^q \overline{z}^p v^{q,p} = h(z)v$$

and is clearly algebraic and satisfies $h(t) = t^k$ for $t \in \mathbb{R}^\times$. □

Definition 2.3. A morphism of Hodge structures of same weight k given by algebraic representations $(V, h_V),\ (W, h_W)$ is an intertwining operator between the representations h_V and h_W.

The above bijection can be made an equivalence of categories if we let the morphisms on the right hand side be maps of $\mathbb{Q}$ vector spaces $f\colon V \to W$ satisfying $f(V^{p,q}) \subseteq W^{p,q}$.

We can also endow the category of Hodge structures with tensor products and duals by using their definition with algebraic representations.

Concretely, for Hodge structures V and W of weights k_V and k_W we have the Hodge structure $V \otimes W$ of weight $k_V + k_W$ given by

$$(V \otimes W)^{p,q} = \sum_{(p_1,q_1)+(p_2,q_2)=(p,q)} V^{p_1,q_1} \otimes W^{p_2,q_2}$$

and the Hodge structure V^* of weight $-k_V$ given by

$$(V^*)^{p,q} = (V^{-p,-q})^*.$$

Example 2.4. We consider the Hodge structure $\mathbb{Q}(n)$ given by

$$\mathbb{Q}(n)^{-n,-n} = \mathbb{Q}, \quad \mathbb{Q}(n)^{p,q} = 0 \text{ for } (p,q) \neq (-n,-n).$$

It is a Hodge structure of weight $-2n$. For a Hodge structure V of weight k, we consider its Tate twists $V(n) := V \otimes \mathbb{Q}(n)$ of weight $k-2n$, so that

$$V(n)^{p,q} = V^{p+n,q+n}.$$

Definition 2.5. The Hodge classes $\mathrm{Hdg}(V,h)$ of a Hodge structure (V,h) are defined by

$$\mathrm{Hdg}(V,h) := \begin{cases} 0 & \text{if} \quad 2 \nmid k, \\ V \cap V^{p,p} & \text{if} \quad k = 2p. \end{cases}$$

Proposition 2.6. *For Hodge structures V and W of same weight k, we have*

$$\mathrm{Hom}_{\mathrm{Hdg}}(V,W) = \mathrm{Hdg}(V^* \otimes W).$$

Proof. Consider a linear map $f\colon V \to W$. It defines a Hodge class in $V^* \otimes W$ iff

$$f(v) = h_W(z) f(h_V(z)^{-1} v) =: ((h_V^*(z) \otimes h_w(z))f)(v)$$

for all $v \in V_{\mathbb{R}}$ and $z \in \mathbb{C}^\times$. It defines a Hodge morphism iff $f(h_V(z)v) = h_W(z)f(v)$ for all $v \in V_{\mathbb{R}}$ and $z \in \mathbb{C}^\times$. Now the claim is clear since both conditions are the same up to changing v to $h(z)v$. □

Definition 2.7. A polarization on a Hodge structure (V,h) is a non-degenerate morphism of Hodge structures $\Psi\colon V \otimes V \to \mathbb{Q}(-k)$ such that $\Psi_{\mathbb{R}}(v, h(i)w)$ is symmetric and positive definite. It can be seen as an isomorphism $V \xrightarrow{\sim} V^*(-k)$ or as a class in $\mathrm{Hdg}(V^* \otimes V^*)$.

Lemma 2.8. *A non-degenerate bilinear map $\Psi\colon V \otimes V \to \mathbb{Q}(-k)$ is a polarization iff*

(1) $\Psi(v,w) = (-1)^k \Psi(w,v)$,
(2) $\Psi_{\mathbb{C}}(v,w) = 0$ *if* $v \in V^{p,q}$ *and* $w \in V^{p',q'}$ *with* $p \neq q'$,
(3) $i^{q-p}\Psi_{\mathbb{C}}(v,\overline{v}) > 0$ *for* $v \in V^{p,q}$ *nonzero.*

Proof. $(\Rightarrow)$: For (1), since Ψ is a morphism of Hodge structures, we have

$$(i\bar{i})^k \Psi(v,w) = \Psi_{\mathbb{R}}(h(i)v, h(i)w) = \Psi_{\mathbb{R}}(w, h(i)^2 v) = \Psi(w, h(-1)v) = (-1)^k \Psi(w,v),$$

where the second equality uses that $\Psi_{\mathbb{R}}(v,h(i)w)$ is symmetric.

For (2), note that

$$(z\overline{z})^k \Psi_{\mathbb{C}}(v,w) = \Psi_{\mathbb{C}}(h(z)v, h(z)w) = \Psi_{\mathbb{C}}(z^p \overline{z}^q v, z^{p'} \overline{z}^{q'} w) = z^{p+p'} \overline{z}^{q+q'} \Psi_{\mathbb{C}}(v,w),$$

and so $\Psi_{\mathbb{C}}(v,w)$ can be nonzero only if $p+p'=q+q'=k$.

For (3), let $v \in V^{p,q}$. From the positive definiteness of $\Psi_{\mathbb{R}}(x,h(i)y)$, we have

$$\begin{aligned} 0 <& \Psi_{\mathbb{R}}((v+\overline{v}), h(i)(v+\overline{v})) \\ =& \Psi_{\mathbb{C}}(v,h(i)v) + \Psi_{\mathbb{C}}(v,h(i)\overline{v}) + \Psi_{\mathbb{C}}(\overline{v},h(i)v) + \Psi_{\mathbb{C}}(\overline{v},h(i)\overline{v}) \\ =& 2i^{q-p}\Psi_{\mathbb{C}}(v,\overline{v}) + i^{p-q}\Psi_{\mathbb{C}}(v,v) + i^{q-p}\Psi_{\mathbb{C}}(\overline{v},\overline{v}), \end{aligned}$$

where the last equality used that $\Psi_{\mathbb{C}}(x,h(i)y)$ is symmetric. If $p \neq q$, then by (2) this is simply $2i^{q-p}\Psi_{\mathbb{C}}(v,\overline{v})$ and the claim follows. If $p=q$, then this is $4 \cdot \Psi_{\mathbb{C}}(v,v)$, and the claim also follows.

$(\Leftarrow)$: First we prove Ψ is a morphism of Hodge structures. For this, it suffices to prove that $\Psi(h(z)v,h(z)w) \overset{?}{=} (z\overline{z})^k\Psi(v,w)$ for $v \in V^{p,q}$ and $w \in V^{p',q'}$. If $p \neq q'$, this is true since both are 0 by (2). If $p=q'$, this means that $p+p'=q+q'=k$, and then

$$\Psi_{\mathbb{C}}(h(z)v,h(z)w) = \Psi_{\mathbb{C}}(z^p\overline{z}^q v, z^{p'}\overline{z}^{q'}w) = z^{p+p'}\overline{z}^{q+q'}\Psi_{\mathbb{C}}(v,w) = (z\overline{z})^k\Psi_{\mathbb{C}}(v,w),$$

Now by (1) and using that Ψ is a morphism of Hodge structures, we have, for $v,w \in V_{\mathbb{R}}$,

$$\Psi_{\mathbb{R}}(v,h(i)w) = (i\overline{i})^k\Psi_{\mathbb{R}}(h(-i)v,w) = (-1)^k\Psi_{\mathbb{R}}(w,h(-i)v) = \Psi_{\mathbb{R}}(w,h(i)v) = \Psi_{\mathbb{R}}(w,h(i)v).$$

Let $v \in V_{\mathbb{R}}$ and write $v = \sum_{p \le k/2} v^p$ where $v^p \in (V^{p,q}+V^{q,p}) \cap V_{\mathbb{R}}$. By (2), we have $\Psi_{\mathbb{R}}(v,h(i)v) = \sum_{p\le k/2}\Psi_{\mathbb{R}}(v^p,h(i)v^p)$. Now choose $w^p \in V^{p,q}$ such that $v^p = w^p + \overline{w^p}$. Now the reverse of the computation done for part (3) above gives us that $\Psi(v,h(i)v) > 0$. □

2.2. Tate structures.

In a similar way, we consider a category of Tate structures.

Definition 2.9. A Tate structure is a $\mathbb{Q}$ vector space V with a continuous action of $G_{\mathbb{Q}} := \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ on $V_{\mathbb{Q}_l}$.

We can also endow the category of Tate structures with tensor products and duals in a natural way.

Example 2.10. We consider the Tate structure $\mathbb{Q}(n)$ with vector space $\mathbb{Q}$ and Galois action χ_l^n where χ_l is the l-adic cyclotomic character. We also let $V(n) := V \otimes \mathbb{Q}(n)$ for any Tate structure V.

Definition 2.11. We say a Tate structure V has (Tate) weight n if for any Frobenius element σ at $p \neq l$, σ^{-1} acts on $V_{\mathbb{Q}_l}$ by a characteristic polynomial with rational coefficients whose roots have complex absolute value $p^{n/2}$.

Remark 2.12. Note that if we have two Tate structures $V \subseteq W$, then if W has weight n, then so does V.

Tate weights also behave well under Tate twists:

Proposition 2.13. *If V is a Tate structure with weight n, then $V(m)$ has weight $n-2m$.*

Proof. Under the cyclotomic character χ_l, a Frobenius at $p \neq l$ goes to multiplication by p. So if σ is a Frobenius element at p, then σ^{-1} acts on $V(m)$ as $\sigma^{-1}\chi_l(\sigma)^{-m} = p^{-m}\sigma^{-1}$ and the claim follows. □

2.3. **The case of projective varieties.** If X is a smooth complex projective variety, we define a bilinear form

$$\Psi \colon \mathrm{H}^{n-k}(X, \mathbb{C}) \otimes \mathrm{H}^{n-k}(X, \mathbb{C}) \to \mathbb{C}$$

by $\Psi(\xi, \eta) = \int_{X(\mathbb{C})} \xi \wedge \eta \wedge \omega^k$ where ω is a Kähler class.

Proposition 2.14. *The bilinear form $(-1)^{\binom{n-k}{2}}\Psi(v, w)$ defines a polarization of $\mathrm{H}^k(X, \mathbb{Q})$.*

Proof. We use Lemma 2.8. That Ψ is non-degenerate and that (1) and (2) hold is clear. (3) follows from the Hodge-Riemann bilinear relations together with the Lefschetz decomposition: The Hodge-Riemann bilinear relations say that for $\xi \in P^{p,q}(X, \mathbb{C})$ a primitive class with $k = p+q$, we have that $i^{q-p}(-1)^{\binom{n-k}{2}}\Psi(\xi, \overline{\xi}) > 0$. By the Lefschetz decomposition, we can decompose

$$\mathrm{H}^{n-k}(X, \mathbb{C}) = \bigoplus_m L^m P^{n-k-2m} = \bigoplus_m L^m \left(\bigoplus_{p+q=n-k-2m} P^{p,q} \right) = \bigoplus_{p+q=n-k} \bigoplus_m L^m P^{p-m,q-m},$$

and since $\Psi(\xi, \eta) = \Psi(L^k\xi, L^k\eta)$ and $(q-m)-(p-m) = q-p$, we have $i^{q-p}(-1)^{\binom{n-k}{2}}\Psi(\xi, \overline{\xi}) > 0$ for all $\eta \in \mathrm{H}^{p,q}(X, \mathbb{Q})$. □

Hence $V := \mathrm{H}^k(X, \mathbb{Q})$ carries a natural polarized Hodge structure. Moreover, if X is defined over $\mathbb{Q}$, V also has a natural Tate structure, since there is a comparison isomorphism $V_{\mathbb{Q}_l} \simeq \mathrm{H}^{n-k}_{ét}(X, \mathbb{Q}_l)$, to étale cohomology, which caries a Galois action of $G_{\mathbb{Q}}$.

2.4. **Hodge, Tate and Weil conjectures.** For a smooth complex projective variety X, we define algebraic cycles to be formal linear combinations of subvarieties, that is, of the form

$$\sum_i c_i Y_i$$

for some finite collection $Y_i \subseteq X$ of subvarieties and $c_i \in \mathbb{Q}$.

For a subvariety $Y \subseteq X$ of codimension i, we can define a cohomology class $[Y] \in \mathrm{H}^{2i}(X, \mathbb{Q})$ to be the dual of its homology class. We extend this to algebraic cycles by linearity. We call cohomology classes of the form $\sum_i c_i[Y_i]$ *algebraic.*

Hodge structures and Tate structures were defined to contextualize the following important conjectures on algebraic geometry.

Conjecture 2.15 (Hodge)**.** *Let X be a smooth complex projective variety. Then all classes in* $\mathrm{Hdg}(\mathrm{H}^{2i}(X, \mathbb{Q}))$ *are algebraic.*

Conjecture 2.16 (Tate)**.** *Let X be a smooth projective variety defined over $\mathbb{Q}$. Then all classes in* $\mathrm{H}^{2i}(X, \mathbb{Q}_l(i))^{G_{\mathbb{Q}}}$ *are algebraic.*

Conjecture 2.17 (Weil)**.** *Let X be a smooth projective variety defined over $\mathbb{Q}$. Then the Tate structure on* $\mathrm{H}^i(X, \mathbb{Q})$ *has weight i.*

In fact, the Tate and Weil conjectures are defined in more generality for varieties over finite fields $\mathbb{F}_q$. The Weil conjectures were proven in full generality by Deligne in 1974.

The Hodge conjecture is known for $i = 1$ and $i = \dim X - 1$ by Lefschetz theorem on $(1,1)$-classes.

3. Kuga-Satake

We assume from now on that (V, h, Ψ) is a Hodge structure of K3-type, that is, a polarized Hodge structure of weight 2 with $\dim V^{2,0} = 1$. We assume also that V carries a Tate structure. Let $Q\colon V \to \mathbb{Q}$ denote the non-degenerate quadratic form on V induced by Ψ. Since Ψ is a polarization, it is negative in $V^{2,0} \oplus V^{0,2}$ and positive in $V^{1,1}$, and so has index $(n-2, 2)$, where $n = \dim V$. We will later apply this to $V = \mathrm{H}^2(X, \mathbb{Q})$ for X a K3 surface.

To (V, Q), one associate its Clifford algebra $C(Q)$. This is a 2^n dimensional associative algebra with the following universal property: for a linear map $f\colon V \to A$ to an algebra A such that $f(v)^2 = Q(v)$, there is a unique morphism of algebras $\tilde{f}\colon C(Q) \to A$ such that f is $V \to C(Q) \xrightarrow{\tilde{f}} A$. It can be defined as the free algebra generated by V, modulo the sub-algebra generated by $vv - Q(v)$.

If $e_1, \ldots, e_n$ is a basis of V such that Q is $\sum d_i X_i^2$ for some $d_i \in \mathbb{Q}$, then $C(Q)$ is an algebra generated by e_i such that $e_i^2 = d_i$ and $e_i e_j + e_j e_i = 0$ for $i \neq j$. It has a sub-algebra given by the even Clifford algebra $C^+(Q)$ of dimension 2^{n-1} spanned by the $e_1^{a_1}\cdots e_n^{a_n}$ with $2 \mid a_1 + \cdots a_n$. Note that $C^+(Q)$ does not depend on the choice of basis.

3.1. **Hodge structure.** We now describe a weight 1 Hodge structure on $C(Q)$ from h. Write $V_{\mathbb{R}} = V_1 \oplus V_2$ with $V_1 \otimes \mathbb{C} = V^{1,1}$ and $V_2 \otimes \mathbb{C} = V^{2,0} \oplus V^{0,2}$. Such decomposition is orthogonal with respect to Q.

Lemma 3.1. *Tthere is a canonical $J \in C^+(Q)_{\mathbb{R}}$ with $J^2 = -1$.*

Proof. Choose a basis f_1, f_2 of V_2 such that $V^{2,0} = \langle f_1 + if_2\rangle$ and $Q(f_1) = -1$. This can be done since Q is negative definite on V_2. Since $Q(V^{2,0}) = 0$, we have $Q(f_1 + if_2) = 0$, hence $Q(f_2) = -1$ and f_1, f_2 are orthogonal. This means that $Q(xf_1 + yf_2) = -(x^2 + y^2)$. Hence $Q(f_1 + f_2) = Q(f_1 - f_2)$, that is, $f_1f_2 + f_2f_1 = 0$.

Now let $J = f_1f_2$. Then $J^2 = f_1f_2f_1f_2 = -f_1^2f_2^2 = -Q(f_1)Q(f_2) = -1$. One can check that J does not depend on the choice of f_1 and f_2. □

Definition 3.2. The weight 1 Hodge structure $(C^+(Q), h_s)$ is given by

$$h_s \colon \mathbb{C}^\times \to \mathrm{GL}(C^+(Q)_{\mathbb{R}}), \quad h_s(a + bi) = a + bJ.$$

We now give $(C^+(Q), h_s)$ a polarization. Write $e^a := e_1^{a_1}\cdots e_n^{a_n}$ for $a = (a_1, \ldots, a_n)$. Also, denote by ι the anti-involution given by $\iota(e^a) = e_n^{a_n}\cdots e_1^{a_1}$.

Lemma 3.3. *We have*

$$\mathrm{Tr}(e^a) = \begin{cases} 0 & \textit{if} \quad a \neq 0, \\ 2^{n-1} & \textit{if} \quad a = 0. \end{cases}$$

In particular, $\mathrm{Tr}(x) = \mathrm{Tr}(\iota(x))$*. Moreover,*

$$\mathrm{Tr}(\iota(e^a)e^b) = \begin{cases} 0 & \textit{if} \quad a \neq b, \\ 2^{n-1}d_1^{a_1}\cdots d_n^{a_n} & \textit{if} \quad a = b. \end{cases}$$

Proof. Considering the matrix of left multiplication of e^a, note that $e^ae^b = \lambda_b e^c$ for some scalar λ_b, and $c \equiv a + b \mod 2$. As $\mathrm{Tr}(e^a) = \sum_{b=c}\lambda_b$, the first claim follow since we have $b = c$ if and only if $a = 0$, and when $a = 0$ we have $\lambda_b = 1$.

Since $\iota(e^a) = \pm e^a$, we have $\iota(e^a)e^b = \lambda e^c$ for some scalar λ and $c \equiv a + b \mod 2$. By the first part, this is nonzero only if $c = 0$, that is, if $a = b$. In this case, we have $\iota(e^a)e^a = d_1^{a_1}\cdots d_n^{a_n}$, and so the claim follows from the first part. □

Recall that Q has signature $(n-2,2)$. So we may assume that $d_1, d_2 < 0$ and $d_3, \dots, d_n > 0$.

Proposition 3.4. *We have that*

$$E\colon C^+(Q) \times C^+(Q) \to \mathbb{Q}, \quad E(v,w) := -\mathrm{Tr}(J\iota(v)w)$$

defines a polarization of $(C^+(Q), h_s)$.

Proof. As $J = f_1 f_2 = -f_2 f_1$, we have $\iota(J) = f_2 f_1 = -J$. Hence

$$E(h_s(z)x, h_s(z)y) = -\mathrm{Tr}(J\iota((a+bJ)x)(a+bJ)y) = -\mathrm{Tr}(J\iota(x)(a^2+b^2)y) = z\bar{z}E(x,y).$$

For the symmetry of $E(x, h_s(i)y)$, we have

$$E(x, h_s(i)y) = -\mathrm{Tr}(J\iota(x)Jy) = -\mathrm{Tr}(\iota(y)(-J)x(-J)) = -\mathrm{Tr}(J\iota(y)h_s(i)x) = E(y, h_s(i)x).$$

Now we prove it is positive definite. As $f_i \in V_2$, we write $f_i = c_i^1 e_1 + c_i^2 e_2$. Then $f_1 f_2 + f_2 f_1 = 0$ amounts to $d_1 c_1^1 c_2^1 + d_2 c_1^2 c_2^2 = 0$ and thus $J = (c_1^1 c_2^2 - c_1^2 c_2^1) e_1 e_2 =: c e_1 e_2$. Note that the scalar c is noznero, as $J \neq 0$. So, writing $x = \sum_i e^{a_i}$, with $a_i = e_1^{a_{i,1}} \dots e_n^{a_{i,n}}$, we have

$$E(x, h_s(i)x) = -\mathrm{Tr}(J\iota(x)Jx) = -\sum_i \sum_j \mathrm{Tr}(\iota(e^{a_i})Je^{a_j}J)$$

By the first part of Lemma 3.3, the only nonzero contribution is for $i = j$, and since $\iota(x)e_i = (-1)^{a_i} e_i \iota(x)$, we have

$$E(x, h_s(i)x) = -\sum_i c^2 \mathrm{Tr}(e_1 e_2 \iota(e^{a_i}) e_1 e_2 e^{a_i}) = \sum_i c^2 (-1)^{a_{i,1}+a_{i,2}} d_1 d_2 \mathrm{Tr}(\iota(e^{a_i})e^{a_i}).$$

By the second part of Lemma 3.3, this is

$$E(x, h_s(i)x) = c^2 d_1 d_2 \sum_i (-d_1)^{a_{i,1}} (-d_2)^{a_{i,2}} d_3^{a_{i,3}} \cdots d_n^{a_{i,n}}.$$

which is > 0 if $x \neq 0$ since each summand is > 0, as $d_1, d_2 < 0$ and $d_3, \dots, d_n > 0$. □

3.2. **Tate structure.** We also note that if V is a Tate structure, then $C^+(Q)$ carries a natural Tate structure coming from V. We define it by $\sigma(e^a) := \sigma(e_1^{a_1}) \cdots \sigma(e_n^{a_n})$ and extended linearly to $C^+(Q)_{\mathbb{Q}_l}$. To see that this is well defined, let A be the free algebra generated by $e_1, \dots, e_n$ and A' the sub-algebra generated by $vv - Q(v)$ for $v \in V$, so that $C(Q) = A/A'$. To define a Galois action on $C(Q)_{\mathbb{Q}_l}$, it suffices to define it on $A_{\mathbb{Q}_l}$ and to prove that $A'_{\mathbb{Q}_l}$ is an invariant subspace.

We have $\sigma(vv-Q(v)) = \sigma(v)\sigma(v)-\sigma(Q(v)) \equiv Q(\sigma(v))-\sigma(Q(v)) \mod A'_{\mathbb{Q}_l}$ and since $Q(\sigma(v)) = \sigma(Q(v))$, we conclude that $\sigma(vv - Q(v)) \in A'_{\mathbb{Q}_l}$. So this is a well defined action on $C(Q)$.

Now it is clear that this action also defines an action on $C^+(Q)_{\mathbb{Q}_l}$, and thus a Tate structure on $C^+(Q)$.

3.3. **Kuga-Satake abelian variety.** The polarized weight 1 Hodge structure $(C^+(Q), h_s, E)$ determines an isogeny class of abelian varieties: consider the dual vector space of $C^+(Q)^{1,0}$

$$W := \left(C^+(Q)^{1,0}\right)^*$$

and we let Γ be a free $\mathbb{Z}$-module with $\Gamma \otimes \mathbb{Q} = (C^+(Q))^*$. Then the projection $\Gamma \subseteq (C^+(Q))^*_{\mathbb{C}} \to W$ is a lattice of W. The quotient is an abelian variety A_Γ with polarization E_Γ determined by E.

Definition 3.5. We call the isogeny class of abelian varieties defined by A_Γ as above the Kuga-Satake isogeny class correspondent to V. If V has an integer structure, the lattice Γ can be chosen canonically, and then the construction defines an abelian variety from V, denoted by $\mathrm{KS}(X)$.

Proposition 3.6. *There is a natural isomorphism of polarized weight one Hodge and Tate structures*

$$(\mathrm{H}^1(A_\Gamma, \mathbb{Q}), E_\Gamma) \simeq (C^+(Q), h_s, E).$$

Proposition 3.7. *Given (V, h, Ψ), there is a morphism $V(1) \hookrightarrow \mathrm{End}(C^+(Q))$ of Hodge and Tate structures depending only on an element $v_0 \in V$ with $Q(v_0) \neq 0$.*

Proof. Consider the map $V(1) \hookrightarrow \mathrm{End}(C^+(Q))$, given by $v \mapsto f_v = (w \mapsto v \cdot w \cdot v_0)$. It is injective since choosing an invertible element v_1 of $C^+(Q)$, we have $f_v(v_1 \cdot v_0) = Q(v_0)(v \cdot v_1)$, and so f_v determines v.

We prove this is a morphism of Hodge structures, that is, that

$$f_{(z\overline{z})^{-1}h(z)v}(w) \stackrel{?}{=} h_s(z) f_v(h_s(z)^{-1}w).$$

Let $z = a + bi$. We want to prove

$$(a^2+b^2)^{-1}h(z)vwv_0 \stackrel{?}{=} (a+bJ)v(a+bJ)^{-1}wv_0.$$

So it suffices to prove

$$h(z)v \stackrel{?}{=} (a+bJ)v(a-bJ).$$

If $v \in V_1$, then v commutes with J and $h(z)v = (a^2 + b^2)$, so this is true. If $v \in V_2$, it suffices to consider $v = f_i$. For $v = f_1$, the right hand side is $(a^2 - b^2)f_1 - 2abf_2$ and for $v = f_2$ it is $2abf_1 + (a^2 - b^2)f_2$. Now, as $f_1 + if_2 \in V^{2,0}$, we have

$$h(z)(f_1 + if_2) = z^2(f_1 + if_2) = ((a^2 - b^2)f_1 - 2abf_2) + i(2abf_1 + (a^2 - b^2)f_2),$$

which amounts to the two equalities we need by separating the real and imaginary parts. □

4. Tate and Weil conjectures for K3 surfaces

The above morphism of Hodge structures allows us to relate the Tate and Weil conjectures of K3 surfaces to the ones of endomorphism of abelian varieties.

The Weil conjecture for abelian varieties, and hence also for the endomorphism of abelian varieties, was proven by Weil in 1948. The Tate conjecture for endomorphism of abelian varieties is a deep result of Faltings from 1983.

Theorem 4.1. *The Tate and Weil conjectures are true for K3 surfaces defined over* $\mathbb{Q}$.

Proof. Let X be a K3 surface defined over $\mathbb{Q}$. The two conjectures for $\mathrm{H}^0(X, \mathbb{Q})$ and $\mathrm{H}^4(X, \mathbb{Q})$ are trivial. Since $\mathrm{H}^1(X, \mathbb{Q}) = \mathrm{H}^3(X, \mathbb{Q}) = 0$, it suffices to consider $V := \mathrm{H}^2(X, \mathbb{Q})$.

Since V has an integral structure given by $\mathrm{H}^2(X, \mathbb{Z})$, we may consider its Kuga-Satake abelian variety $A := \mathrm{KS}(X)$. By Proposition 3.6 and Proposition 3.7, we have a morphism of Hodge and Tate structures $V(1) \overset{\varphi}{\hookrightarrow} \mathrm{End}(\mathrm{H}^1(A, \mathbb{Q}))$.

Since the Weil conjecture is true for $\mathrm{H}^1(A, \mathbb{Q})$, it is also true for its endomorphism space. Hence the right hand side has weight 0 as a Tate structure and so $V(1)$ also has weight 0 by Remark 2.12. Hence $V = \mathrm{H}^2(X, \mathbb{Q})$ has weight 2, by Proposition 2.13, and so its Weil conjecture is true.

For the Tate conjecture, let $\xi \in \mathrm{H}^2(X, \mathbb{Q}_l(1))^{G_\mathbb{Q}} = V_{\mathbb{Q}_l}(1)^{G_\mathbb{Q}}$ be a Galois equivariant element. Since the Tate conjecture is true for $\mathrm{H}^1(A, \mathbb{Q})$, $\varphi(\xi)$ is induced by a self-isogeny of A. Hence $\varphi(\xi) \in \mathrm{End}(\mathrm{H}^1(A, \mathbb{Q}_l))^{0,0}$ and, since φ is an injective morphism of Hodge structures, we must also have $\xi \in \mathrm{H}^2(X, \mathbb{Q}_l)(1)^{0,0} = \mathrm{H}^2(X, \mathbb{Q}_l)^{1,1}$. As the Hodge conjecture is known for divisors, ξ is the class induced by a divisor D with $\mathbb{Q}_l$ coefficients. By the Galois equivariance, such divisor can be found with $\mathbb{Q}$ coefficients. □

I pledge my honor that this paper represents my own work in accordance with University regulations.