FUNDAMENTAL LEMMA (PROJECT FOR 18.748, FALL 2020)

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Abstract. This note discusses the statement of the Fundamental Lemma, together with the necessary background on endoscopy.

Contents

1. Introduction

The Fundamental Lemma is a conjectural identity between linear combinations of orbital integrals of certain pairs (G, H) of reductive groups over a p-adic field. Via the Arthur–Selberg trace formula, orbital integrals are related to the representation theory of the group, and thus the Fundamental Lemma amounts to correspondences between the representation theory of the groups G and H. Very roughly, we may picture these relations as follows

$$\begin{array}{ccc} \text{representation theory of } G & \xleftrightarrow{\text{trace formula}} & \text{orbital integrals on } G \\ \updownarrow{\scriptstyle \text{Langlands conjectures}} & & \updownarrow{\scriptstyle \text{Fundamental Lemma}} \\ \text{representation theory of } H & \xleftrightarrow{\text{trace formula}} & \text{orbital integrals on } H \end{array}$$

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The fundamental lemma arises in this way from conjectural relations in the representation theory, and is in turn used to prove some of these relations. The fundamental lemma is currently not proved in its full generality, but particular cases have been very important in the theory of automorphic representations, being critical parts of proofs of results such as the Base Change for GL_2 and the local Langlands conjecture for GL_n.

2. Unramified reductive groups

Let F be a p-adic field, that is, either a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_{p^n}((t))$ for some n. We will consider connected reductive algebraic groups G over F.

Recall that G is *split* if it has a maximal torus T which is isomorphic over F to a product of $\mathbb{G}_m$. G is *quasi-split* if it has a Borel subgroup over F.

Definition 2.1. A connected reductive linear algebraic group G is an *unramified reductive group* if G is quasi-split and if there is a unramified extension F'/F such that $G \times_F F'$ is split over F'.

Recall that in the case that G is split, it is classified by its *root datum* $(X^*, X_*, \Phi, \Phi^\vee)$ where X^*, X_* are the character and cocharacter groups of a maximal torus of G, and $\Phi \subseteq X^*$ and $\Phi^\vee \subseteq X_*$ are the roots and coroots.

If G is split over an unramified extension F'/F, then we have an action of $\mathrm{Gal}(F'/F)$ on $G \times_F F'$, which induces an action on X^*. Since F'/F is unramified, $\mathrm{Gal}(F'/F) = \langle \sigma \rangle$ where σ is the Frobenius automorphism, and we have an induced map $\sigma\colon X^* \to X^*$. Naturally, σ induces a bijection on Φ.

If furthermore G is quasi-split, then σ takes positive roots to positive roots. This is because a set of positive roots is determined by a choice of Borel, which is defined over F. So in this case we get a bijection $\sigma\colon \Phi^+ \to \Phi^+$.

In short, for an unramified reductive group G, we associated the data $(X^*, X_*, \Phi, \Phi^\vee, \sigma)$. These in fact classify such unramified reductive groups:

Theorem 2.2. *Unramified reductive groups over F are classified by $(X^*, X_*, \Phi, \Phi^\vee, \sigma)$, where $(X^*, X_*, \Phi, \Phi^\vee)$ is a root datum and $\sigma\colon \Phi^+ \to \Phi^+$ is an automorphism of finite order.*

Proof. Let $\tilde{G}$ be the split group over F with the given root datum and $\tilde{B} \to \tilde{G}$ a Borel. Then unramified quasi-split forms of $\tilde{G}$ are classified by $\mathrm{H}^1\left(\mathrm{Gal}(F^{ur}/F), \mathrm{Aut}(\tilde{B} \to \tilde{G})\right)$. That is, they are classified by the action of Frobenius $\sigma\colon \Phi_G^+ \to \Phi_G^+$, and the continuity imposes that σ is of finite order. □

3. Endoscopic groups

Let G be an unramified reductive group over F with classifying data $(X^*, X_*, \Phi_G, \Phi_G^\vee, \sigma_G)$.

Definition 3.1. An unramified reductive group H over F is an *unramified endoscopic group* of G if its classifying data has the form

$$(X^*, X_*, \Phi_H, \Phi_H^\vee, \sigma_H)$$

satisfying the following: there is $s \in \mathrm{Hom}(X_*, \mathbb{C}^\times)$ and $w \in W(\Phi_G)$ such that

(a) $\Phi_H^\vee = \{\alpha \in \Phi_G^\vee \colon s(\alpha) = 1\}$,
(b) $\sigma_H = w \circ \sigma_G$,
(c) $\sigma_H(s) = s$.

Next, we aim to prove that if H is unramified endoscopic of G, then any maximal tori T_H of H is isomorphic to a maximal tori T_G of G.

Given a torus T over F, let F'/F be a finite extension such that T splits over F'. Then $\mathrm{Gal}(F'/F)$ acts on $T \times_F F'$, and hence on $X^*(T)$, giving a morphism $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}(X^*(T))$ with finite image.

Theorem 3.2. *A torus T over F is classified by a triple (X^*, X_*, ρ) where $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}(X^*)$ has finite image.*

Moreover, given G an unramified reductive group with classifying data $(X^*(T), X_*(T), \Phi_G, \Phi_G^\vee, \sigma_G)$, then T embeds over F as a maximal torus in G if and only if the following two conditions hold:

(a) The image of ρ is contained in $W(\Phi_G) \rtimes \langle \sigma_G \rangle$.
(b) The composition $\mathrm{Gal}(\overline{F}/F) \xrightarrow{\rho} W(\Phi_G) \rtimes \langle \sigma_G \rangle \to \langle \sigma_G \rangle$ is the same as $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(F^{\mathrm{ur}}/F) \to \langle \sigma_G \rangle$ where Frobenius is mapped to σ_G.

Proof. Let $\tilde{T}$ be the split torus with the given root data. Then its forms are classified by $\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), \mathrm{Aut}(\tilde{T})\right) = \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), \mathrm{Aut}(X^*)\right) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\overline{F}/F), \mathrm{Aut}(X^*))$. Here, continuous means that it factors through a finite extension, and hence that has a finite image.

For the second part, again let $\tilde{G}$ be the split form of G, and $\iota\colon \tilde{T} \to \tilde{G}$ be a maximal torus. Now the possible T that embed as a maximal torus in G are classified by the preimage of σ_G in

$$\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), \mathrm{Aut}(\tilde{T} \to \tilde{G})\right) \to \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), \mathrm{Aut}(\tilde{G})\right).$$

Since we may identify the above map as

$$\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), W(\Phi_G) \rtimes \mathrm{Aut}(\Phi_G)\right) \to \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), \mathrm{Aut}(X^*, \Phi_G)\right),$$

we obtain the description we want. Note that we are using that, since σ_G preserves positive roots, the natural map $W(\Phi_G) \rtimes \langle\sigma_G\rangle \to \mathrm{Aut}(X^*)$ is injective. □

Now if H is an unramified endoscopic group of G, the above theorem says that the conditions for a torus to embed maximally onto H are more restrictive than to embed on G. So we have the following corollary.

Corollary 3.3. *Let H be an unramified endoscopic group of G. Then every maximal torus T_H of H is isomorphic to a maximal torus T_G of G, and T_G can be chosen such that $X_*(T_G) \simeq X_*(T_H)$ and $X^*(T_G) \simeq X^*(T_H)$ as Galois modules.*

4. Stable conjugacy

Recall that two elements $g_1, g_2 \in G(F)$ are called *stably conjugate* if they are conjugate over $\overline{F}$. We will denote $g_1 \sim_F g_2$ for F conjugacy and $g_1 \sim g_2$ for stable conjugacy.

We will soon consider summations over stable conjugates modulo F-conjugates, that is, summations over the set

$$C(g) := \{\gamma \in G(F)\colon g \sim \gamma\}/\sim_F.$$

Although this will only be used in the computations made in the examples, it is helpful to investigate the difference between $\sim$ and $\sim_F$ better, and to describe what $C(g)$ is in terms of Galois cohomology.

Lemma 4.1. *Let $g \in G(F)$. Let $H = C_G(g)$. Then the set of F-conjugates of g is in bijection with $G(F)/H(F)$, and the set of stable conjugates of g is in bijection with $(G/H)(F)$. Moreover, we have the following exact sequence of pointed sets*

$$0 \to C(g) \to \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), H(\overline{F})\right) \to \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), G(\overline{F})\right).$$

Proof. The first two bijections are given as follows: consider the maps

$$G(F)/H(F) \to G(F) \quad \gamma H(F) \mapsto \gamma g \gamma^{-1},$$
$$(G/H)(\overline{F}) \to G(\overline{F}) \quad \gamma H(\overline{F}) \mapsto \gamma g \gamma^{-1}.$$

The first map gives the first bijection, and the second map gives a bijection between $(G/H)(\overline{F})$ and stable conjugates of g in $G(\overline{F})$. By definition, $(G/H)(F)$ is the preimage of $G(F)$ in the second map, and thus we have the desired bijection.

Now consider the exact sequence of schemes

$$0 \to H \to G \to G/H \to 0.$$

Taking Galois cohomology, we obtain the exact sequence of pointed sets

$$0 \to G(F)/H(F) \to (G/H)(F) \to \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), H(\overline{F})\right) \to \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), G(\overline{F})\right).$$

Since $G(F)$ acts on the first two terms and acts transitively on the first, it makes sense to consider the cokernel of the first map: it is the quotient of this action. This cokernel can be identified with $C(g)$ by the definition of $C(g)$. Thus, we obtain the exact sequence we want

$$0 \to C(g) \to \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), H(\overline{F})\right) \to \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), G(\overline{F})\right).$$ □

5. Strongly regular elements

Definition 5.1. For G a reductive group over F, we say that a semisimple element $\gamma \in G(F)$ is *strongly regular* if its centralizer is a maximal torus.

Let γ_0 be a fixed strongly regular element of $G(F)$ with centralizer T, and $\gamma' \in G(F)$ a stable conjugate of γ_0. To such a γ', we will associated an element of the Galois cohomology $\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), T(\overline{F})\right)$ as follows: If $g \in G(\overline{F})$ is such that $\gamma' = g^{-1}\gamma_0 g$ and given $\tau \in \mathrm{Gal}(\overline{F}/F)$, then $a_\tau := \tau(g)g^{-1}$ centralizes γ_0. Indeed,

$$a_\tau \gamma_0 = \tau(g)g^{-1}\gamma_0 = \tau(g)\gamma' g^{-1} = \tau(g\gamma' g^{-1})\tau(g)g^{-1} = \tau(\gamma_0)a_\tau = \gamma_0 a_\tau.$$

Thus $a_\tau \in T(\overline{F})$, and this gives a function $a\colon \mathrm{Gal}(\overline{F}/F) \to T(\overline{F})$. It is a 1-cocycle since

$$a_{\tau\sigma} = \tau(\sigma(g))g^{-1} = \tau(\sigma(g))\tau(g)^{-1}\tau(g)g^{-1} = \tau(a_\sigma)a_\tau.$$

We also note that a is also continuous, since the conjugation of γ_0 and γ' happens over some finite extension of F, that is, g is defined over some finite extension. Hence $a \in \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), T(\overline{F})\right)$.

Moreover, one can explicitly compute such cohomology group.

Theorem 5.2 (Tate–Nakayama)**.** *Let F'/F be an unramified extension that splits T. Then*

$$\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), T(\overline{F})\right) \simeq \frac{\{u \in X_* \colon \sum_{\tau \in \mathrm{Gal}(F'/F)} \tau u = 0\}}{\{\tau v - v \colon \tau \in \mathrm{Gal}(F'/F),\ v \in X_*\}}.$$

Proof. The right hand side can be identified with $\hat{\mathrm{H}}^{-1}(\mathrm{Gal}(\overline{F}/F), X_*)$, where $\hat{\mathrm{H}}$ denotes Tate cohomology. Now since T splits over F' which is unramified over F, by Hilbert 90 and inflation-restriction, these cohomologies are isomorphic to

$$\mathrm{H}^1\left(\mathrm{Gal}(F'/F), T(F')\right) \quad \text{and} \quad \hat{\mathrm{H}}^{-1}(\mathrm{Gal}(F'/F), X_*).$$

Note that if E/F, is any extension, then $\mathrm{H}^1\left(\mathrm{Gal}\left(E/F\right), E^\times\right) = 0$ by Hilbert 90, and note that if E/F is finite unramified, then $\mathrm{H}^1\left(\mathrm{Gal}\left(E/F\right), E^\times\right)$ is cyclic of order $[E : F]$ for all E/F unramified: this is because $0 \to U_E \to E^\times \xrightarrow{\mathrm{val}} \mathbb{Z} \to 0$ and because U_E has trivial cohomology for $i \geq 1$ since E/F is unramified. So $\mathrm{H}^2\left(E/F, E^\times\right) \simeq \mathrm{H}^2\left(E/F, \mathbb{Z}\right)$, which is cyclic of order $[E : F]$ if E/F is cyclic.

Hence, by Tate's theorem we obtain the isomorphism

$$\hat{\mathrm{H}}^{-1}(\mathrm{Gal}(F'/F), X_*) \xrightarrow{\sim} \mathrm{H}^1\left(\mathrm{Gal}(F'/F), (F')^\times \otimes X_*\right) = \mathrm{H}^1\left(\mathrm{Gal}(F'/F), T(F')\right),$$

which is given by cup product with a generating class of $\mathrm{H}^2\left(\mathrm{Gal}(F'/F), (F')^\times\right)$. ☐

Now if H is an unramified endoscopic group of G and $\gamma \in H(F)$ is strongly regular, then its centralizer T_H is isomorphic to some maximal torus $T_G \subseteq G$ by the above corollary. Fix such an isomorphism $T_H \xrightarrow{\sim} T_G$. The data for H includes an element $s \in \mathrm{Hom}(X_*, \mathbb{C}^\times)$, and since $\sigma_H(s) = s$, this s in fact factors through the co-invariants of X_*. Thus, from the above description of $\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), T(\overline{F})\right)$, s induces a homomorphism

$$\kappa \colon \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), T(\overline{F})\right) \to \mathbb{C}^\times.$$

For $\gamma_0, \gamma' \in G(F)$ stably conjugate, we denote $\kappa(\gamma_0, \gamma') = \kappa(a)$ where $a \in \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), T(\overline{F})\right)$ was defined above.

With this preparation, we can finally define the orbital integrals we will be comparing.

Definition 5.3. Let G be an unramified reductive group over F, and H an unramified endoscopic group of G. Let $\gamma \in H(F)$ be strongly regular and $T_H = C_H(\gamma)$. Consider an isomorphism $T_H \xrightarrow{\sim} T_G \subseteq G$ and assume that $\gamma \mapsto \gamma_0 \in G(F)$ where γ_0 is also strongly regular. In such case, we say γ is *G-strongly regular.* Let K_G, K_T be hyperspecial maximal compact subgroups of G and T_G. We consider the following combination of orbital integrals

$$\Lambda_{G,H}(\gamma) := \left| \prod_{\alpha \in \Phi_G} (\alpha(\gamma_0) - 1) \right|^{1/2} \frac{\mathrm{vol}_T(K_T)}{\mathrm{vol}_G(K_G)} \sum_{\gamma' \in C(\gamma_0)} \kappa(\gamma_0, \gamma') \mathcal{O}_{\gamma'}(1_{K_G}).$$

This is called a κ-orbital integral. We also consider the stable orbital integral

$$\Lambda_H^{st}(\gamma) := \left| \prod_{\alpha \in \Phi_H} (\alpha(\gamma) - 1) \right|^{1/2} \frac{\mathrm{vol}_T(K_T)}{\mathrm{vol}_H(K_H)} \sum_{\gamma' \in C(\gamma)} \mathcal{O}_{\gamma'}(1_{K_H}).$$

Remark 5.4. The value of $\Lambda_{G,H}(\gamma)$ actually depends on the choice of $T_H \xrightarrow{\sim} T_G \subseteq G$, but only up to a root of unity. There is a way to define a *transfer factor* $\Delta(\gamma, \gamma_0)$ so that $\Delta(\gamma, \gamma_0)\Lambda_{G,H}(\gamma)$ is well defined. I will not attempt to define the transfer factor in general, and will implicitly assume that the choice of isomorphism is such that $\Delta(\gamma, \gamma_0) = 1$, which is always possible to be done.

6. The fundamental lemma

Conjecture 6.1 (Fundamental Lemma)**.** *Let G be an unramified reductive group over F, and H an unramified endoscopic group of G. Let $\gamma \in H(F)$ be G-strongly regular. Then*

$$\Lambda_{G,H}(\gamma) = \Lambda_H^{st}(\gamma).$$

This is part of the more general context of *matching*. The Transfer Conjecture predicts that on the same situation above and given $f \in C_c^\infty(G(F))$, there is $f^H \in C_c^\infty(H(F))$ that does not depend on γ such that the following two quantities are equal:

$$\Lambda_{G,H}(\gamma, f) := \left| \prod_{\alpha \in \Phi_G} (\alpha(\gamma_0) - 1) \right|^{1/2} \frac{\mathrm{vol}_T(K_T)}{\mathrm{vol}_G(K_G)} \sum_{\gamma' \in C(\gamma_0)} \kappa(\gamma_0, \gamma') \mathcal{O}_{\gamma'}(f),$$

$$\Lambda_H^{st}(\gamma, f^H) := \left| \prod_{\alpha \in \Phi_H} (\alpha(\gamma) - 1) \right|^{1/2} \frac{\mathrm{vol}_T(K_T)}{\mathrm{vol}_H(K_H)} \sum_{\gamma' \in C(\gamma)} \mathcal{O}_{\gamma'}(f^H).$$

In this case, we say that f^H matches f. The Fundamental Lemma is the assertion that 1_{K_H} matches 1_{K_G}.

7. Examples

For concreteness, we will look at some examples of pairs of groups G, H satisfying the hypothesis of the fundamental lemma, and we will unravel the definitions of the two orbital integrals.

7.1. SL_2 **and** $U_E(1)$**.** First consider $G = \mathrm{SL}_2$. Its classifying data if $(\mathbb{Z}, \mathbb{Z}, \{\pm 2\}, \{\pm 1\}, 1)$. Let w be the nontrivial reflection and $s \in \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^\times)$ given by $s(n) = (-1)^n$. Then $\Phi_H = \emptyset$, and so H has classifying data given by $(\mathbb{Z}, \mathbb{Z}, \emptyset, \emptyset, w)$, which gives the 1-dimensional torus $U_E(1)$ split by an unramified quadratic extension $E = F(\sqrt{\epsilon})$. By the Tate–Nakayama isomorphism,

$$\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), U_E(1)\right) \simeq \frac{\{u \in \mathbb{Z} \colon u + wu = 0\}}{\{wv - v \colon v \in \mathbb{Z}\}} = \frac{\mathbb{Z}}{\{-v - v \colon v \in \mathbb{Z}\}} = \mathbb{Z}/2\mathbb{Z}.$$

and κ is the the morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$ given by $\kappa(n) = (-1)^n$. The matching of maximal tori is given by

$$U_E(1) \hookrightarrow \mathrm{SL}_2, \quad \gamma := a + b\sqrt{\epsilon} \mapsto \gamma_0 := \begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix}$$

and $\gamma = a + b\sqrt{\epsilon}$ is G-strongly regular when $b \neq 0$.

Note that $\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), \mathrm{SL}_2(\overline{F})\right) = 0$, since we have an exact sequence $0 \to \mathrm{SL}_2 \to \mathrm{GL}_2 \xrightarrow{\det} \mathrm{GL}_1 \to 0$, and by Hilbert 90, GL_n has vanishing first cohomology, so we may conclude the vanishing of the first cohomology of SL_2 as the determinant map $\mathrm{GL}_2(F) \to \mathrm{GL}_1(F)$ is surjective. Hence, $C(\gamma_0) \simeq \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), U_E(1)\right) \simeq \mathbb{Z}/2\mathbb{Z}$ and so there are two stable conjugacy classes of γ_0 modulo F-conjugacy, namely

$$\gamma_0 = \begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix}, \quad \gamma' := \begin{pmatrix} a & b \\ b\epsilon & a \end{pmatrix}$$

Then $\kappa(\gamma_0, \gamma_0) = 1$ and $\kappa(\gamma_0, \gamma') = -1$. So assuming that the measures are normalized so that K_G, K_H, K_T have measure 1, we have

$$\Lambda_{G,H}(\gamma) = \pm|\gamma - \overline{\gamma}|^{1/2}(\mathcal{O}_{\gamma_0}(1_{K_G}) - \mathcal{O}_{\gamma'}(1_{K_G}))$$

and

$$\Lambda_H^{st}(\gamma) = \mathcal{O}_\gamma(1_{K_H}).$$

The $\pm$ in $\Lambda_{G,H}(\gamma)$ is the transfer factor, which we will not specify.

In this particular case, one can prove the fundamental lemma by computing all the above orbital integrals using Bruhat–Tits trees.

7.2. **Unramified unitary groups.** Let E/F be an unramified quadratic extension. We consider the unitary groups $U_E(n) = \{X \in M_{n\times n}(E) \colon {}^t\overline{X}X = I\}$. Its classifying data is $(\mathbb{Z}^n, \mathbb{Z}^n, \Phi, \Phi^\vee, \sigma)$ where $\Phi = \{e_i - e_j \colon i \neq j\}$ and $\sigma(e_i) = -e_i$ for all i. For H, we take $w = 1$ and $s \in \mathrm{Hom}(X^*, \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^n$ be given by $(1, \dots, 1, -1, \dots, -1)$ with n_1 terms equal to 1 and n_2 terms equal to -1, with $n_1 + n_2 = n$. Now it is easy to see that $\Phi_H = \{\alpha \in \Phi_G \colon s(\alpha) = 1\}$ are precisely the roots of the form $e_i - e_j$ with $i \neq j$ and either $i, j \leq n_1$ or $i, j > n_1$. Thus $H = U_E(n_1) \times U_E(n_2)$ is the endoscopic group of G we are considering. Note that H is a subgroup of G.

The stable conjugacy classes of unitary groups are all represented by diagonal elements, so let $T \subseteq H \subseteq G$ be the diagonal torus, and let $\gamma \in T$ be G-strongly regular. Since $T \simeq U_E(1)^n$, by the computation using Tate–Nakayama above, we have that

$$\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), T(\overline{F})\right) = (\mathbb{Z}/2\mathbb{Z})^n.$$

If $a = (a_1, \dots, a_n) \in (\mathbb{Z}/2\mathbb{Z})^n$, then the character κ acts as $\kappa(a) = (-1)^{a_{n_1+1} + \cdots + a_{n_1+n_2}}$.

For concreteness, let's look at the example $n_1 = n_2 = 1$. Let $\gamma = \mathrm{diag}(\gamma_1, \gamma_2) \in T(F)$ be a G-strongly regular element. This means that $\gamma_1 \neq \gamma_2$.

Now we show that $C(\gamma)$ has order 2. In general, since $U_E(n)$ splits over E, using inflation restriction one can prove that $\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), U_E(n)(\overline{F})\right) = \mathrm{H}^1\left(\mathrm{Gal}(E/F), U_E(n)(E)\right)$. Using cocycles, we can thus see that

$$\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), U_E(n)(\overline{F})\right) \simeq \frac{\{A \in \mathrm{GL}_n(E) \colon A = {}^t\overline{A}\}}{\{A{}^t\overline{A} \colon A \in \mathrm{GL}_n(E)\}},$$

and so $\mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), U_E(1)\right) \simeq F^\times/\mathrm{Nm}_{E/F}E^\times \simeq \mathbb{Z}/2\mathbb{Z}$ has representatives 1 and π for an uniformizer π on E. The map

$$(F^\times/\mathrm{Nm}_{E/F}E^\times) \times (F^\times/\mathrm{Nm}_{E/F}E^\times) = \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), U_E(1) \times U_E(1)\right) \to \mathrm{H}^1\left(\mathrm{Gal}(\overline{F}/F), U_E(2)\right),$$

which has kernel $C(\gamma)$, is given by

$$(a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Explicitly writing the conditions for such matrix to be of the form $A{}^t\overline{A}$ for $A \in \mathrm{GL}_n(E)$, one can compute that the elements of the kernel are $(1,1)$ and (π, π), so that $C(\gamma)$ has order 2.

Hence, there are two stable conjugacy classes of γ modulo F-conjugacy, namely

$$\gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \quad \gamma' := \begin{pmatrix} \gamma_1 + \beta & \alpha\beta \\ \overline{\alpha}\beta & \gamma_2 - \beta \end{pmatrix}.$$

where $\beta = \frac{\gamma_2 - \gamma_1}{\pi}$ and $\alpha \in E^\times$ is such that $\alpha\overline{\alpha} = \pi - 1$. As one should expect, $\kappa(\gamma, \gamma) = 1$ and $\kappa(\gamma, \gamma') = -1$.

Again assuming that the measures are normalized such that K_G, K_H, K_T have measure 1, putting all the above together we have

$$\Lambda_{G,H}(\gamma) = \pm \frac{|\gamma_1 - \gamma_2|}{|\gamma_1\gamma_2|^{1/2}} (\mathcal{O}_\gamma(1_{K_G}) - \mathcal{O}_{\gamma'}(1_{K_G}))$$

and

$$\Lambda_H^{st}(\gamma) = \mathcal{O}_{\gamma_1}(1_{K_{U_E(1)}})\mathcal{O}_{\gamma_2}(1_{K_{U_E(1)}}).$$