p-ADIC MODULAR FORMS À LA KATZ

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Abstract. These are notes from a talk given at STAGE about Chapter 2 of Katz's *p-adic properties of modular schemes and modular forms.*

Contents

1. The Hasse invariant

In the treatment of p-adic modular forms à la Katz, the modular form E_{p-1} played an important role since its q-expansion was 1 modulo p. We begin by defining the *Hasse invariant*, a Katz modular form over $\mathbb{F}_p$ that will play a similar role.

Let R be a ring with $p = 0$. Let E/R be an elliptic curve and ω be a basis for $\underline{\omega}_{E/R}$. This determines a dual basis $\eta \in H^1(E, \mathcal{O}_E)$ by Serre duality. We note that the absolute Frobenius $F\colon E \to E$ induces a map $F^*\colon H^1(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E)$.

Definition 1.1 (Hasse invariant)**.** For E, ω as above, we define the *Hasse invariant* $A(E, \omega)$ such that $F^*(\eta) = A(E, \omega) \cdot \eta$.

Remark 1.2. Note that if R is a finite field, then $A(E, \omega) = 0$ if and only if E is supersingular.

Proposition 1.3. *$A(E, \omega) \in M(\mathbb{F}_p, 1, p-1)$ is a (Katz) modular form over $\mathbb{F}_p$ of weight $p-1$ and level* 1.

Proof. For $\lambda \in R^\times$, if $\omega' = \lambda\omega$ then $\eta' = \lambda^{-1}\eta$, and so

$$F^*(\lambda^{-1}\eta) = \lambda^{-p}F^*(\eta) = \lambda^{-p}A(E, \omega) \cdot \eta = \lambda^{1-p}A(E, \omega) \cdot (\lambda^{-1}\eta),$$

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and hence $A(E, \lambda\omega) = \lambda^{1-p} A(E, \omega)$. □

Proposition 1.4. *$A \in S(\mathbb{F}_p, 1, p-1)$ is a holomorphic (Katz) modular form over $\mathbb{F}_p$ of weight $p-1$ and level 1. Moreover, its q-expansion is 1.*

Proof. We use that $H^1(E, \mathcal{O}_E) = \mathrm{Lie} E$ is the R-module of invariant derivations of E/R, and the action of F^* on $\eta \in H^1(E, \mathcal{O}_E)$ consists of taking the p-th iterate $F^*(\eta) = \eta^{\circ(p)}$.

The completion of the Tate curve along its identity section is $\hat{\mathbb{G}}_m$, and one can check that the canonical differential ω_{can} is simply $\mathrm{d}t/t$. For the uniformizing parameter $t_0 = t - 1$, this becomes $\omega_{\mathrm{can}} = \mathrm{d}t_0/(1+t_0)/$ This means that the invariant derivation η_{can} dual to ω_{can} satisfies $\eta_{\mathrm{can}}(t_0) = 1 + t_0$. Now note $\eta_{\mathrm{can}}^{\circ(p)}(t_0) = 1 + t_0$, and so $F^*(\eta_{\mathrm{can}})$ agrees with η_{can} at t_0. This is enough to conclude that $F^*(\eta_{\mathrm{can}}) = \eta_{\mathrm{can}}$. □

In the study of (Katz) p-adic modular forms, we will use an appropriate lift of A to other coefficient rings.

Definition 1.5. We say that n is p-good if there exist a holomorphic (Katz) modular form $\tilde{A} \in S(\mathbb{Z}[1/n], n, p-1)$ such that $A = \tilde{A} \mod p$.

Proposition 1.6. *The following n are p-good.*

p	n
2	$3 \leq n \leq 11,\ 2 \nmid n$
3	$n \geq 2,\ 3 \nmid n$
≥ 5	$n \geq 1,\ p \nmid n$

Proof. This follows from the base-change theorems. □

We want to define a suitable notion of p-adic modular forms in the style of Katz. If we hope that some form of the q-expansion principle to hold, then $\tilde{A}^{p^{N-1}}$ should "converge" to 1 as $N \to \infty$, since its q expansions are 1 modulo p^N. However, this is not consistent where $\tilde{A}(E, \omega) = 0$, i.e. at "supersingular" curves. So, for such a notion of p-adic modular forms, we will need to consider our test objects away from such supersingular locus.

2. p-adic modular forms with growth conditions

Let R_0 be a p-adically complete ring and a $\mathbb{Z}[1/n]$-module. For any $r \in R_0$ and a p-good n, we will define the module $M(R_0, r, n, k)$ of p-adic modular forms with growth condition r. Roughly, we restrict the possible test objects away from modular forms whose Hasse invariant is in a (p-adic) disk of radius $|r|_p$ around 0.

If $r = 1$, then we are restricting to the ordinary locus, and turns out that we will recover Serre's modular forms (at least for integral weights k). Taking r to have positive p-adic valuation will correspond to *overconvergent* modular forms.

Definition 2.1. A (Katz) *p-adic modular form* $f \in M(R_0, r, n, k)$ over R_0 of weight k, level n and growth r is a rule that assigns to a triple $(E/S, \alpha_n, Y)$ where

(a) E/S is an elliptic curve over an R_0-scheme S on which p is nilpotent,
(b) α_n is a level n structure,
(c) Y is a section of $\underline{\omega}^{\otimes 1-p}$ such that $Y \cdot \tilde{A}(E/S, \alpha_n) = r$,

a section $f(E/S, \alpha_n, Y) \in \underline{\omega}_{E/S}^{\otimes k}$ such that (i) f only depends on the S-isomorphism class of such triples and (ii) f commutes with base change of R_0-schemes.

Remark 2.2. As in the definition of Katz modular forms, we may consider f as a rule from $(E/R, \omega, \alpha_n, Y)$ to R, for R_0-algebras R with p nilpotent and $Y \in R$ with $Y \cdot \tilde{A}(E, \omega, \alpha_n) = r$ compatible with extension of scalars and such that

$$f(E/R, \lambda\omega, \alpha_n, \lambda^{p-1}Y) = \lambda^{-k} f(E/R, \omega, \alpha_n, Y).$$

Remark 2.3. We may also evaluate f for p-adically complete R_0-algebras R by passing to the limit.

Definition 2.4. We say $f \in M(R_0, r, n, k)$ is holomorphic at infinity and denote $f \in S(R_0, r, n, k)$ if for all $N \geq 1$ and level n structures α_n the value

$$f(\mathrm{Tate}(q^n), \omega_{\mathrm{can}}, \alpha_n, r \cdot (\tilde{A}(\mathrm{Tate}(q^n), \omega_{\mathrm{can}}, \alpha_n))^{-1}) \in (R_0/p^N R_0) \otimes_{\mathbb{Z}[1/n]} \mathbb{Z}[1/n, \zeta_n]((q))$$

lies in $(R_0/p^N R_0) \otimes_{\mathbb{Z}[1/n]} \mathbb{Z}[1/n, \zeta_n][\![q]\!]$.

Note that, formally, we have

$$M(R_0, r, k, n) = \varprojlim M(R_0/p^N R_0, r, k, n) \quad \text{and} \quad S(R_0, r, k, n) = \varprojlim S(R_0/p^N R_0, r, k, n).$$

Our first goal will be to understand a bit more about these spaces. We first do so in the case that p is nilpotent in R_0, by giving it a moduli interpretation.

2.1. **When p is nilpotent in R_0.** For now, we assume p is nilpotent in R_0. Let $n \geq 3$ be p-good. We denote $\mathscr{L} := \underline{\omega}^{\otimes 1-p}$.

We consider the moduli problem

$$\mathcal{F}_{R_0,r,n} \colon S \mapsto \{S\text{-isomorphism classes of } (E/S, \alpha_n, Y)\}.$$

By the definition of M_n, this is the same as

$$\mathcal{F}_{R_0,r,n} \colon S \mapsto \{R_0\text{-morphisms } g \colon S \to M_n \otimes R_0, \text{ together with } Y \in g^*\mathscr{L} \text{ satisfying } Y \cdot g^*\tilde{A} = r\}.$$

This is a subfunctor of the following functor

$$\mathcal{F}_{R_0,n} \colon S \mapsto \{R_0\text{-morphisms } g \colon S \to M_n \otimes R_0, \text{ together with } Y \in g^*\mathscr{L}\}.$$

Geometrically, this last functor should be representable by the "total space" of $\mathscr{L}$. The next proposition makes this precise.

Proposition 2.5. *$\mathcal{F}_{R_0,n}$ is representable by $\operatorname{Spec}_{M_n\otimes R_0}(\operatorname{Sym}^\bullet(\check{\mathscr{L}}))$. Moreover, $\mathcal{F}_{R_0,r,n}$ is representable by $\operatorname{Spec}_{M_n\otimes R_0}(\operatorname{Sym}^\bullet(\check{\mathscr{L}})/(\tilde{A}-r))$.*

Proof. The first statement follows essentially from the definition of the scheme $\operatorname{Spec}_{M_n\otimes R_0}(\operatorname{Sym}^\bullet(\check{\mathscr{L}}))$. Let us work this out and see how to integrate the growth condition.

Given a morphism $g \colon S \to M_n \otimes R_0$, cover $M_n \otimes R_0$ by open sets $\operatorname{Spec} B_i$ such that $\mathscr{L}$ trivializes with invertible section $\check{l}_i$. Cover $g^{-1}(\operatorname{Spec} B_i)$ by open sets $\operatorname{Spec} A_{ij}$. By the definition of the relative spectrum, we have $\operatorname{Spec}_{M_n\otimes R_0}(\operatorname{Sym}^\bullet(\check{\mathscr{L}}))|_{\operatorname{Spec} B_i} = \operatorname{Spec}(B_i[\check{l}_i])$.

Given $Y \in g^*\mathscr{L}$, we can lift the morphism $g \colon B_i \to A_{ij}$ to $\tilde{g}_{ij} \colon B_i[\check{l}_i] \to A_{ij}$ by

$$\sum_k b_k(\check{l}_i)^k \mapsto \sum_k b_k(Y \cdot g^*\check{l}_i)^k.$$

One can check that these glue to define $\tilde{g} \colon S \to \operatorname{Spec}_{M_n\otimes R_0}(\operatorname{Sym}^\bullet(\check{\mathscr{L}}))$. Moreover, the data necessary for such a lift is exactly the data of giving such a $Y \in g^*\mathscr{L}$. This means that $\mathcal{F}_{R_0,n}$ is represented by $\operatorname{Spec}_{M_n\otimes R_0}(\operatorname{Sym}^\bullet(\check{\mathscr{L}}))$.

If we want to specify a growth condition, then having $Y \cdot \tilde{A} = r$ is the same as $\tilde{g}_{ij}$ factoring through $B_i[\check{l}_i]/(\tilde{A}-r)$ for all i, j. So $\mathcal{F}_{R_0,r,n}$ is represented by $\operatorname{Spec}_{M_n\otimes R_0}(\operatorname{Sym}^\bullet(\check{\mathscr{L}})/(\tilde{A}-r))$. $\square$

As a consequence of this, we have the following description of (Katz) p-adic modular forms from (Katz) modular forms of varying weights.

Corollary 2.6. *We have* $M(R_0, r, n, k) = \left(\bigoplus_{j\geq 0} M(R_0, n, k + j(p-1))\right)/(\tilde{A} - r)$.

Proof. We have $M(R_0, r, n, k) = H^0(\mathrm{Spec}_{M_n\otimes R_0}(\mathrm{Sym}^\bullet(\check{\mathscr{L}})/(\tilde{A} - r)), \underline{\omega}^{\otimes k})$, and this is

$$H^0(M_n \otimes R_0, \underline{\omega}^{\otimes k} \otimes \mathrm{Sym}^\bullet(\check{\mathscr{L}})/(\tilde{A} - r)) = H^0(M_n \otimes R_0, \bigoplus_{j\geq 0} \underline{\omega}^{k+j(p-1)}/(\tilde{A} - r)).$$

Since $M_n \otimes R_0$ is affine, this is

$$\left(H^0(M_n \otimes R_0, \bigoplus_{j\geq 0} \underline{\omega}^{k+j(p-1)})\right)/(\tilde{A} - r) = \left(\bigoplus_{j\geq 0} M(R_0, n, k + j(p-1))\right)/(\tilde{A} - r). \qquad \square$$

The subspace of holomorphic forms is what one would expect.

Proposition 2.7. *The submodule* $S(R_0, r, n, k) \subseteq M(R_0, r, n, k)$ *corresponds to*

$$H^0(\mathrm{Spec}_{\overline{M_n}\otimes R_0}(\mathrm{Sym}^\bullet(\check{\mathscr{L}})/(\tilde{A} - r), \underline{\omega}^{\otimes k}).$$

Proof. Essentially, this follows since the cusps on $\overline{M_n}$ are ordinary, i.e., $\tilde{A}$ is invertible on the cusps.

We may assume $\zeta_n \in R_0$. Let $\infty = \overline{M_n} \otimes R_0 - M_n \otimes R_0$. We have seen that completion of $\overline{M_n} \otimes R_0$ has its ring isomorphic to a finite number of copies of $R_0[\![q]\!]$. The completion of $\mathrm{Spec}_{\overline{M_n}\otimes R_0}(\mathrm{Sym}^\bullet(\check{\mathscr{L}})/(\tilde{A}-r))$ along the inverse image of ∞, hence, is a finite number of copies of

$$R_0[\![q]\!][Y]/(Y \cdot \tilde{A}(\mathrm{Tate}(q^n), \omega_{\mathrm{can}}, \alpha_n) - r).$$

Since $\tilde{A}(\mathrm{Tate}(q^n), \omega_{\mathrm{can}}, \alpha_n)$ is invertible, this ring is equal to $R_0[\![q]\!]$.

So similarly as before, a section of $H^0(\mathrm{Spec}_{M_n\otimes R_0}(\mathrm{Sym}^\bullet(\check{\mathscr{L}})/(\tilde{A} - r), \underline{\omega}^{\otimes k})$ can be extended to ∞ exactly when it is holomorphic. $\square$

Corollary 2.8. *We have* $S(R_0, r, n, k) = H^0(\overline{M_n} \otimes R_0, \bigoplus_{j\geq 0} \underline{\omega}^{k+j(p-1)}/(\tilde{A} - r))$.

Proof. This is as in the previous corollary. $\square$

2.2. **When R_0 is p-adically complete.**

Definition 2.9. We say n is p, k-good if (i) n is p-good, (ii) $3 \leq n \leq 11$ if $k = 1$ or $k = 0, p = 2$.

Theorem 2.10. *Let $n \geq 3$ be p, k-good. Let $r \in R_0$ be not a zero divisor. Then we have an isomorphism*

$$\left(\varprojlim_N H^0(\overline{M_n}, \bigoplus_{j\geq 0} \underline{\omega}^{k+j(p-1)}) \otimes_{\mathbb{Z}[1/n]} (R_0/p^N R_0)\right)/(\tilde{A}-r) \xrightarrow{\sim} \varprojlim_N S(R_0/p^N R_0, r, n, k) = S(R_0, r, n, k).$$

Concretely, this means that elements of $S(R_0, r, n, k)$ *are formal sums* $\sum_{a\geq 0} s_a$ *where we have* $s_a \in S(R_0, n, k + a(p-1))$ *and* $s_a \to 0$ *when* $a \to \infty$, *where we identify two of these sums if they differ by a multiple of* $(\tilde{A} - r)$.

Proof for $k > 0$. Let $\mathscr{S} = \bigoplus_{j\geq 0} \underline{\omega}^{k+j(p-1)}$ on $\overline{M_n}$. Put $\mathscr{S}_N = \mathscr{S} \otimes R_0/p^n R_0$ and consider the exact sequence

$$0 \to \mathscr{S}_N \xrightarrow{\tilde{A}-r} \mathscr{S}_N \to \mathscr{S}_N/(\tilde{A} - r) \to 0.$$

We saw in the proof of the base change theorems that $H^1(\overline{M_n}, \underline{\omega}^{\otimes l}) = 0$ if $l \geq 2$ or if $l = 1$ and $3 \leq n \leq 11$. So if n is p, k-good and $k > 0$, we have $H^1(\overline{M_n}, \mathscr{S}_N) = 0$.

By the base change theorems, we then get an inverse system

$$0 \to H^0(\overline{M_n}, \mathscr{S}) \otimes R_0/p^N R_0 \xrightarrow{\tilde{A}-r} H^0(\overline{M_n}, \mathscr{S}) \otimes R_0/p^N R_0 \to H^0(\overline{M_n}, \mathscr{S}_N/(\tilde{A} - r)) \to 0.$$

Since the transition maps for the left most modules are surjective, taking inverse limit is exact, and we get what we want.

For $k = 0$, we have instead that $H^1(\overline{M_n}, \mathscr{S}) = H^1(\overline{M_n}, \mathcal{O})$ so one has to be more careful. □

3. Basis of p-adic modular forms

By the previous result, we can describe (Katz) p-adic modular forms in terms of regular (Katz) modular forms and multiplications by $\tilde{A}$. To find a basis of p-adic modular forms, we first analyze the multiplication by $\tilde{A}$ more closely.

Lemma 3.1. *Let* $n \geq 3$ *be* p*-good. Then the map*

$$S(\mathbb{Z}_p, n, k) \xrightarrow{\cdot \tilde{A}} S(\mathbb{Z}_p, n, k + p - 1)$$

is injective, and its cokernel is a finite free $\mathbb{Z}_p$*-module.*

Proof. The injectivity follows from the q-expansion principle since $\tilde{A}$ has q-expansions are invertible. We need to prove that the above cokernel is torsion free. It suffices to prove that $\tilde{A}f \in p \cdot S(\mathbb{Z}_p, n, k + p - 1) \implies f \in p \cdot S(\mathbb{Z}_p, n, k)$. Again, this follows from the q-expansion principle since the q-expansions of $\tilde{A}$ are invertible. □

This means that we can choose once and for all a direct sum decomposition

$$H^0(\overline{M_n} \otimes \mathbb{Z}_p, \underline{\omega}^{k+(j+1)(p-1)}) \simeq \tilde{A} \cdot H^0(\overline{M_n} \otimes \mathbb{Z}_p, \underline{\omega}^{k+j(p-1)}) \oplus B(n, k, j + 1).$$

Setting $B(n,k,0) := H^0(\overline{M_n} \otimes \mathbb{Z}_p, \underline{\omega}^k)$ and $B(R_0,n,k,j) = B(n,k,j) \otimes_{\mathbb{Z}_p} R_0$ give us a direct sum decomposition

$$\bigoplus_{a=0}^{j} B(R_0,n,k,a) \xrightarrow{\sim} S(R_0,n,k+j(p-1)), \quad \sum_{a=0}^{j} b_a \mapsto \sum_{a=0}^{j} \tilde{A}^{j-a} b_a.$$

Definition 3.2. Let $B^{\text{rigid}}(R_0,n,k)$ be the R_0-module consisting of formal sums $\sum_{a\geq 0} b_a$ for $b_a \in B(R_0,n,k,a)$ such that $b_a \to 0$, that is, such that for any $N > 0$ there exist $M > 0$ for which $b_a \in p^N B(R_0,n,k,a)$ for all $a \geq M$.

Theorem 3.3. *Let $n \geq 3$ be p,k-good. Let $r \in R_0$ be not a zero divisor. Then we have an isomorphism*

$$\phi_r \colon B^{\text{rigid}}(R_0,n,k) \xrightarrow{\sim} S(R_0,r,n,k)$$

given by

$$\sum_{a\geq 0} b_a \mapsto \left((E/S,\alpha_n,Y) \mapsto \sum_{a\geq 0} b_a(E/S,\alpha_n)\cdot Y^a \right).$$

We may think of the image of $\sum_{a\geq 0} b_a$ as "$\sum_{a\geq 0} r^a \frac{b_a}{\tilde{A}^a}$."

Proof. Given $\sum_{a\geq 0} s_a$ with $s_a \in S(R_0,n,k+a(p-1))$, consider the decomposition $s_a = \sum_{i+j=a} \tilde{A}^i b_{j,a}$ with $b_{j,a} \in B(R_0,n,k,j)$. Note that $b_{j,a} \to 0$ as $a \to \infty$ uniformly in j. Then,

$$\begin{aligned}\sum_{a\geq 0} s_a &= \sum_{a\geq 0}\sum_{i+j=a} \tilde{A}^i b_{j,a} \\ &= \sum_{a\geq 0}\sum_{i+j=a} r^i b_{j,a} + (\tilde{A}-r)\left(\sum_{a\geq 0}\sum_{i+j=a} b_{j,a} \sum_{u+v=i-1} \tilde{A}^u r^v\right)\end{aligned}$$

so as elements of $S(R_0,r,n,k)$, we have $\sum_{a\geq 0} s_a = \sum_{a\geq 0}\sum_{i+j=a} r^i b_{j,a}$. Then

$$\sum_{a\geq 0} s_a = \sum_{a\geq 0}\sum_{i+j=a} r^i b_{j,a} = \sum_{i\geq 0}\sum_{j\geq 0} r^i b_{j,i+j} = \sum_{i\geq 0} r_i \sum_{j\geq 0} b_{j,i+j}.$$

Letting $b_i' := \tilde{A}^i \sum_{j\geq 0} b_{j,i+j}$, we then have $\sum_{a\geq 0} s_a = \phi_r(\sum_{i\geq 0} b_i')$.

For injectivity, it suffices to prove that if $\phi_r(\sum_{a\geq 0} b_a) = 0$, then $b_a \in p^N B(R_0,n,k,a)$ for all N. For a fixed N, there is M large such that $\phi_r(\sum_{a=0}^{M} b_a) \equiv 0 \mod p^N$ and $b_a \equiv 0 \mod p^N$ for $a > M$. Then $\sum_{a=0}^{M} r^a \tilde{A}^{M-a} b_a \equiv 0 \mod p^N$, and hence we condlude that $r^a b_a \in p^N B(R_0,n,k,a)$ for all N,a. Since r is not a zero divisor, this means $b_a = 0$. □

Corollary 3.4. *Consider the map* $S(R_0, r, n, k) \to S(R_0, 1, n, k)$ *defined modularly by* $(E/S, \alpha_n, Y) \mapsto (E/S, \alpha_n, rY)$*. This is injective, and we have a commutative diagram*

$$\begin{array}{ccc} B^{\mathrm{rigid}}(R_0, n, k) & \xrightarrow{\phi_{r,1}} & B^{\mathrm{rigid}}(R_0, n, k) \\ \downarrow{\scriptstyle \phi_r} & & \downarrow{\scriptstyle \phi_1} \\ S(R_0, r, n, k) & \longrightarrow & S(R_0, 1, n, k) \end{array}$$

where $\phi_{r,1}\colon \sum_{a\geq 0} b_a \mapsto \sum_{a \geq 0} r^a b_a$.

4. $r = 1$ AND THE q-EXPANSION PRINCIPLE

When $r = 1$, we obtain a q-expansion principle for (Katz) p-adic modular forms by reducing to the q-expansion principle we have seen before.

Theorem 4.1. *Let* $n \geq 3$ *be* p, k*-good. Let* $x \in R_0$ *be a divisor of* p^N *for some* $N > 0$*. For* $f \in S(R_0, 1, n, k)$*, the following are equivalent:*

(1) $f \in x \cdot S(R_0, 1, n, k)$,

(2) the q*-expansions of* f *all lie in* $x \cdot R_0[\zeta_n][\![q]\!]$,

(3) on each of the $\varphi(n)$ *connected components of* $\overline{M_n} \otimes \mathbb{Z}[1/n, \zeta_n]$*, there is at least one cusp where the* q*-expansion of* f *lies in* $x \cdot R_0[\zeta_n][\![q]\!]$.

Proof. It suffices to prove (3) $\implies$ (1). Let $f = \phi_1(\sum_{a\geq 0} b_a)$. If $M > 0$ is such that $b_a \in p^N \cdot B(R_0, n, k, a)$ for $a > M$, then we may assume without loss of generality that $f = \phi_1(\sum_{a=0}^M b_a)$. Now the q-expansion of f at $(\mathrm{Tate}(q^n), \omega_{\mathrm{can}}, \alpha_n, \tilde{A}^{-1})$ is the same as that of

$$\sum_{a=0}^M b_a \tilde{A}^{-a} = \frac{\sum_{a=0}^M b_a \tilde{A}^{M-a}}{\tilde{A}^M}.$$

Hence $\sum_{a=0}^M b_a \tilde{A}^{M-a} \in S(R_0, n, k + M(p-1))$ has q-expansion lying in $x \cdot R_0[\zeta_n][\![q]\!]$ in at least one cusp in each component. By the regular q-expansion principle, this means that $\sum_{a=0}^M b_a \tilde{A}^{M-a} \in x \cdot S(R_0, n, k + M(p-1))$, and hence $b_a \in x \cdot B(R_0, n, k, a)$ for all a. This implies that $f \in x \cdot S(R_0, 1, n, k)$. □

Now we see that we recover Serre's definition, at least when the weight is integral.

Proposition 4.2. *Let* $n \geq 3$ *be* p, k*-good. Suppose that for each cusp* α *of* $\overline{M_n} \otimes \mathbb{Z}[\zeta_n]$ *we are given a power series* $f_\alpha \in R_0[\zeta_n][\![q]\!]$*. Then the following are equivalent:*

(1) The f_α *are the* q*-expansions of an element* $f \in S(R_0, 1, n, k)$.

(2) For every $N > 0$, there is an integer $M \equiv 0 \mod p^{N-1}$ and a modular form $g_N \in S(R_0, n, k + M(p-1))$ whose q-expansions are congruent modulo p^N to the given f_α.

Proof. (1) $\Longrightarrow$ (2): As in the proof of the theorem above, for every M sufficiently large there is $g \in S(R_0, n, k, M(p-1))$ such that the q-expansions of f agree with the q-expansions of $g\tilde{A}^{-M}$. We choose $M \equiv 0 \mod p^{N-1}$. Since we have the q expansion congruences $\tilde{A}(q) \equiv 1 \mod p$, from $p^{N-1} \mid M$ we obtain $\tilde{A}^M(q) \equiv 1 \mod p^N$. We conclude that the q-expansions as f and g agree modulo N.

(2) $\Longrightarrow$ (1): Possibly multiplying g_N by a power of $\tilde{A}^{p^{N-1}}$, we may assume their weights $k+M_N(p-1)$ are increasing. Then $(g_{N+1} - g_N\tilde{A}^{M_{N+1}-M_N})$ has q-expansion a multiple of p^N. Hence $(g_{N+1} - g_N\tilde{A}^{M_{N+1}-M_N}) \in p^N S(R_0, n, k + M_{N+1}(p-1))$. Then $f = \sum_{N\geq 1}(g_{N+1} - g_N\tilde{A}^{M_{N+1}-M_N})$ is a well defined element of $S(R_0, 1, n, k)$ that agree with the q-expansions of g_N modulo p^N. □

Remark 4.3. Similarly as in Borys's talk, we can obtain analogues to the above results for $n = 1, 2$ (still requiring that n be p, k-good) by considering the space of such forms of level n inside the space of forms of larger levels, and mapping down via projectors.